Name...SOLUTIONS...
I.D. number...

Math 2280-1
Second Midterm
April 2, 2010

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Find all possible solutions $y = y(t)$ to the differential equation

$$D^{(3)}(y)(t) - D(y)(t) = 3 + 4\,e^{-t}.$$

(20 points)

$$y''' - y' = 3 + 4e^{-t}$$

y_H: $y = e^{rx} \rightarrow p(r) = r^3 - r = 0$

$r(r-1)(r+1) = 0$

$$\boxed{y_H = c_1 + c_2 e^{+t} + c_3 e^{-t}}$$

y_{P_1} for $L(y_{P_1}) = 3$ guess $y_{P_1} = ct$ ($\lambda = 0$ alg. mult 1)

$L(y_{P_1}) = -c$ so take $\boxed{y_{P_1} = -3t}$

$= 3$; $c = -3$

y_{P_2} for $L(y_{P_2}) = 4e^{-t}$

try $y_{P_2} = kte^{-t}$ ($\lambda = -1$ alg mult 1)

$y_P' = (k - kt)e^{-t}$

$y_P'' = (-k - k + kt)e^{-t}$

$= (-2k + kt)e^{-t}$

$y_P''' = (2k + k - kt)e^{-t}$

$= (3k - kt)e^{-t}$

$L(y_P) = y_P''' - y_P' = ke^{-t}[3 - \cancel{t} - 1 + \cancel{t}]$

$[3 - 1]$

$= 2ke^{-t}$ $k = 2$ $\boxed{y_{P_2} = 2te^{-t}}$

$= 4e^{-t}$

$$\boxed{y = y_{P_1} + y_{P_2} + y_H = 2te^{-t} - 3t + c_1 + c_2 e^{+t} + c_3 e^{-t}}$$

2a) Let A be the matrix

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}.$$

Find the eigenvalues of A, and corresponding eigenvectors. Work carefully, because you'll be using this matrix in several later problems. (Hint: you should get -1 and -4 for the eigenvalues.)

(10 points)

$$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+3)(\lambda+2)-2$$
$$= \lambda^2+5\lambda+4$$
$$= (\lambda+4)(\lambda+1)$$

$\lambda=-1$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda=-4$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

2b) If "k" is any non-zero scalar, what are the eigenvalues and eigenvectors of the scalar multiple of A given by

$$k\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}?$$

(Hint: If **v** is an eigenvector of A, compute kA**v**.)

(5 points)

if $A\vec{v} = \lambda\vec{v}$

then $kA\vec{v} = (k\lambda)\vec{v}$

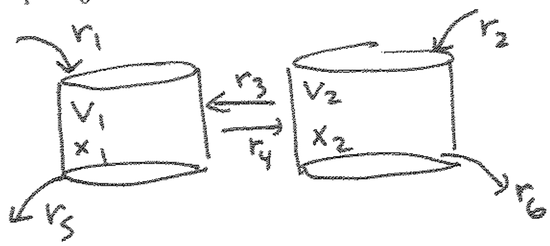
so eigenvalues get multiplied by k, eigenvectors stay the same

for our A, then for kA:

$\lambda=-k$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda=-4k$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

3) Consider the following input-output model, represented by two tanks with two input pipes, two connecting pipes, and two output pipes. Each pipe moves water at a corresponding rate (volume/time), indicated by the six rates $r_1 \ldots r_6$.



3a) What two equations should the six rate constants satisfy so that the volume in each tank remains constant? $r_i = r_o$: tank 1 : $r_1 + r_3 = r_4 + r_5$ (4 points)

tank 2 : $r_2 + r_4 = r_3 + r_6$

3b) Consider two interconnected delta estuary ponds, which can be studied using this input output model. Each pond has average daily volume of $V_1 = V_2 = 100$ acre feet of water, which we will treat as constant. Each pond has its own small freshwater stream, one flowing into the lower pond at a rate of $r_1 = 1$ acre feet per day, and another flowing into the upper pond, also at a rate of $r_2 = 1$ acre feet of water per day. Because of tidal action water flows from the lower pond to the upper pond at a rate of $r_4 = 1$ acre feet per day, and from the upper back to the lower pond at $r_3 = 2$. All water leaves the upper pond through the outlet to the lower one, so $r_6 = 0$. Water leaves the lower pond through an outlet to the ocean at rate $r_5 = 2$. Sometimes salt is introduced into this system because of ocean flooding. In this case, let $x_1(t)$ be the amount of salt (in tons) in the lower pond, and let $x_2(t)$ be the amount in the upper pond. Show that the rates of change of salt amounts (tons/day) are modeled by the first order system

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

You should begin by checking that the daily average volumes in each pond remain constant. (6 points)

$r_1 = 1$
$r_2 = 1$
$r_3 = 2$
$r_4 = 1$
$r_5 = 2$
$r_6 = 0$

$r_1 + r_3 = 3 = r_4 + r_5$ ✓
$r_2 + r_4 = 2 = r_3 + r_6$
& so $V_1 = V_2 = \text{const.}$

$$x_1' = -\frac{(r_4+r_5)}{\cancel{V_1}}\frac{x_1}{V_1} + r_3\frac{x_2}{V_2}$$
$$= -\frac{3}{100}x_1 + \frac{2}{100}x_2$$

$$x_2' = r_4\frac{x_1}{V_1} - (r_3+r_6)\frac{x_2}{V_2}$$
$$= 1\cdot\frac{x_1}{100} - \frac{2}{100}x_2$$

agrees with displayed system

3c) Notice the matrix in the system (3b) is a multiple of the one in (2). Use your previous work there to write down the general solution to the system in (3b).

(8 points)

for $B = \frac{1}{100}\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$

$\lambda = -\frac{1}{100} = -.01$ $\qquad$ $\lambda = -\frac{4}{100} = -.04$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\vec{x}_H(t) = c_1 e^{-.01t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-.04t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

3d) After a storm there is 1 ton of salt in the lower pond, and no salt in the upper pond. Denoting this time by t=0 (days), solve the initial value problem for this first order system of differential equations in (3b), to get formulas for the amounts of salt in each pond at day t.

(8 points)

$$\vec{x}' = .01\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}\vec{x}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

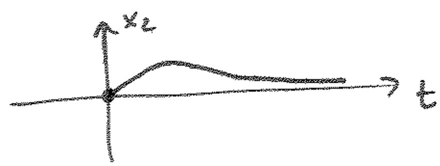
so $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{3}e^{-.01t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-.04t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

3e) Give a qualitative description of the behavior of the functions $x_1(t)$ and $x_2(t)$ for positive t.

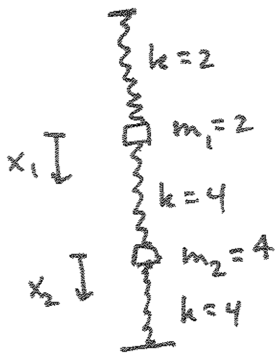
(4 points)

$x_1(t) = \frac{1}{3}\left(e^{-.01t} + 2e^{-.04t}\right)$ decays exponentially, from $x_1(0) = 1$

$x_2(t) = \frac{1}{3}\left(e^{-.01t} - e^{-.04t}\right)$ is initially 0, increases to a max value, and then decays exponentially



4a) Consider the hanging system of three springs and two masses as indicated below. Hanging at rest, spring and gravity forces are in balance for certain equilibrium positions of the two masses. Measuring vertical displacements from these equilibria as indicated, and using the indicated mass and spring constant values, show that if these masses are set into vertical motion then their displacements from equilibrium satisfy the second order system



$$\begin{bmatrix} \dfrac{d^2 x_1}{dt^2} \\ \dfrac{d^2 x_2}{dt^2} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(5 points)

$$2x_1'' = -2x_1 + 4(x_2 - x_1) = -6x_1 + 4x_2$$
$$4x_2'' = -4(x_2 - x_1) - 4x_2 = 4x_1 - 8x_2$$

so
$$x_1'' = -3x_1 + 2x_2$$
$$x_2'' = x_1 - 2x_2$$
✓

4b) What is the general solution to this second order system of differential equations? (Hint: you've seen this matrix before.)

(5 points)

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$$

$\lambda = -1$ $\quad\quad$ $\lambda = -4$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad\quad$ $\vec{v} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\omega = \sqrt{1} = 1$ $\quad\quad$ $\omega = \sqrt{4} = 2$

$$\vec{x}_H(t) = (c_1 \cos t + c_2 \sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 \cos 2t + c_4 \sin 2t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

4c) Describe the two fundamental modes of this system.

(5 points)

slow mode $\quad C_1 \cos(t - \alpha_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ masses oscillate in phase with identical amplitudes. ($\omega = 1$ rad/sec)

fast mode $\quad C_2 \cos(2t - \alpha_2)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad$ masses oscillate twice as fast ($\omega = 2$), out of phase (in opposite directions), and the first mass is oscillating with twice the amplitude of the second.

5) Find the matrix exponential e^{Bt} for the matrix

$$B=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

(20 points)

power series:

$$B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = I + R$$

$$e^{Bt} = e^{It+Rt} = e^{t} I e^{Rt} = e^{t} e^{Rt}$$

since $e^{It+Rt} = e^{It}e^{Rt}$ (R & I commute).

$$R = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = Rot_{\pi/2}.$$

$$R^2 = -I = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$R^3 = -R$$

$$R^4 = I$$

so $$e^{Rt} = I + t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + \dots$$

$$= \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}.$$

so $$\boxed{e^{Bt} = e^{t}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}}$$

or,

FMS : $\begin{vmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix} = (\lambda-1)^2+1 = (\lambda-1+i)(\lambda-1-i)$ $\qquad \lambda = 1 \pm i$

$\lambda = 1+i$:

$$\left[\begin{array}{cc|c}-i & -1 & 0\\ 1 & -i & 0\end{array}\right]$$

$$\left[\begin{array}{cc|c}i & 1 & 0\\ 1 & -i & 0\end{array}\right]$$

$$\begin{array}{r}\\ +iR_1+R_2\end{array}\left[\begin{array}{cc|c}i & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$\vec{v} = \begin{bmatrix}-1\\ i\end{bmatrix}$$

$$e^{\lambda t}\vec{v} = e^{(1+i)t}\begin{bmatrix}-1\\ i\end{bmatrix} = e^{t}(\cos t + i\sin t)\begin{bmatrix}-1\\ i\end{bmatrix} = \begin{bmatrix}-e^{t}\cos t & -ie^{t}\sin t\\ -e^{t}\sin t & +ie^{t}\cos t\end{bmatrix}$$

$= \vec{u}(t) + i\vec{v}(t)$, $\{-\vec{u}(t), \vec{v}(t)\}$ is a basis for $\vec{x}' = B\vec{x}$

take $$\Phi(t) = \begin{bmatrix}e^{t}\cos t & -e^{t}\sin t\\ e^{t}\sin t & e^{t}\cos t\end{bmatrix}$$

note $\Phi(0) = I$, so

$$e^{Bt} = \Phi(t)\Phi(0)^{-1} = \Phi(t)$$