Name __Solutions__

Student I.D. _____

Math 2280-1
Exam #1
February 19, 2010

    Please show all work for full credit. This exam is closed book and closed note. You may use a
scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra
equations. **In order to receive full or partial credit on any problem, you must show all of your
work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are
indicated in the right-hand margin. **Good Luck!**

Score

POSSIBLE

1_____ 25

2_____ 25

3_____ 20

4_____ 20

5_____ 10

TOTAL_____ 100

1) Consider a brine tank which holds 1000 gallons of continuously-mixed liquid. Let x(t) be the amount of salt (in pounds) in the tank at time t. The in-flow and out-flow rates are both 200 gallons/hour, and the concentration of salt flowing into the tank is 0.05 pounds per gallon.

1a) Use the information above and your modeling ability to derive the differential equation for x(t):

$$\frac{dx}{dt} = 10 - 0.2 \cdot x$$

(5 points)

$r_i = 200 \text{ g/m}$ $c_i = .05 \text{ lb/g}$

$V = 1000 \text{ g}$
$x(t) -$

$r_o = 200 \text{ g/m}$

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

$$= (200)(.05) - 200 \left(\frac{x}{1000}\right) \quad \text{g/min} \cdot \text{lb/g}$$

$$\boxed{\frac{dx}{dt} = 10 - .2x}$$

1b) Solve the initial value problem for this differential equation assuming there are initially 30 pounds of salt in the tank.

$$x(0) = 30$$

(10 points)

$$x' + .2x = 10$$

$$e^{.2t}(x' + .2x) = 10e^{.2t}$$

$$(e^{.2t}x)' = 10e^{.2t}$$

$$e^{.2t}x = 50e^{.2t} + C$$

$$x = 50 + Ce^{-.2t}$$

$$x(0) = 30 = 50 + C \implies C = -20$$

$$\boxed{x(t) = 50 - 20e^{-.2t}}$$

1c) Your father has a "sporty" car of mass 1000 kg. The engine and transmission of his car is able to provide a maximum force of 10,000 Newtons (kg m/sec^2) at every speed. Assume the drag forces on the car from air and the road are proportional to the car's velocity. What is the drag coefficient (and its units) if the differential equation for the car's velocity reduces to

$$\frac{dv}{dt} = 10 - 0.2 \cdot v \ ?$$

(4 points)

$$m v' = \text{net forces}.$$

$$1000 \frac{dv}{dt} = 1000 \left(10 - .2v \right)$$
$$= 10,000 - 200v$$

so the drag coefficient is $\boxed{200 \ \text{kg/sec}}$ = Newton-sec/m.

(because multiplying by m/sec yields Newtons)

1d) Notice that you've ended up with a differential equation equivalent to the one (1a). Suppose you're at the raceway and driving your Dad's car at 30 m/sec. If you begin accelerating at the maximum rate possible, i.e. according to the differential equation above, then how long will it be until you're going 45 m/sec (about 101 miles/hour)? Note that you've already solved the appropriate initial value problem!

(4 points)

$$v(0) = 30$$

$$v(t) = 50 - 20 e^{-.2t}$$

solve $45 = 50 - 20 e^{-.2t}$

$$20 e^{-.2t} = 5$$

$$e^{-.2t} = \frac{1}{4}$$

$$-.2t = -\ln 4 \ ; \quad t = \frac{\ln 4}{.2} \ \text{sec}.$$

1e) What's the fastest your Dad's car can go?

(2 points)

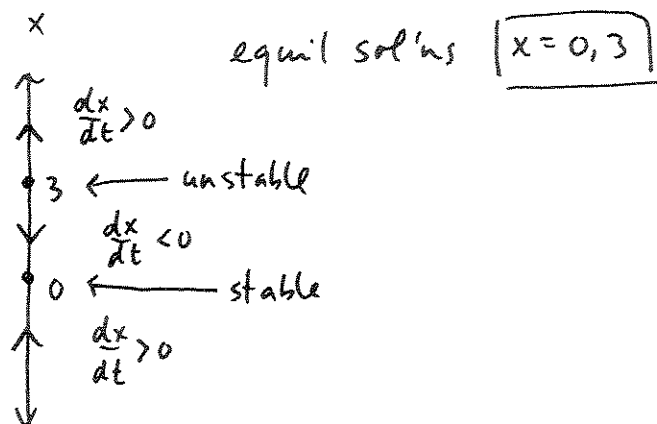$$50 \quad m/sec.$$

is the limiting velocity

2) Consider the differential equation

$$\frac{dx}{dt} = x^2 - 3 \cdot x .$$

2a) Find the equilibrium solutions for this differential equation and use a phase portrait to determine their stability.

(8 points)

$$x(x-3)$$

equil sol'ns $\boxed{x = 0, 3}$

x

$\frac{dx}{dt} > 0$

$3 \longleftarrow$ unstable

$\frac{dx}{dt} < 0$

$0 \longleftarrow$ stable

$\frac{dx}{dt} > 0$

2b) If this differential equation was modeling a population, would it be a logistic or doomsday/extinction model? Briefly explain your answer.

(4 points)

doomsday-extinction: $x_0 > 3 \implies$ quadratic pop. growth rate $\implies$ finite time to $\lim\limits_{t \to t_0} x(t) = +\infty$

$x_0 < 3 \implies x(t) \to 0.$

2c) Solve the initial value problem for this differential equation, with $x(0) = 4$. (There's extra space on the next page.)

(10 points)

$$\frac{dx}{x(x-3)} = dt$$

$$\int \frac{1}{3}\left(\frac{1}{x-3} - \frac{1}{x}\right) = \int dt$$

$$\frac{1}{3} \ln\left|\frac{x-3}{x}\right| = t + C_1$$

$$\ln\left|\frac{x-3}{x}\right| = 3t + C_2$$

$$\left|\frac{x-3}{x}\right| = e^{C_2} e^{3t}$$

$$\frac{x-3}{x} = C e^{3t}$$

@ $t = 0$: $\frac{4-3}{4} = C = \frac{1}{4}$

$$\frac{x-3}{x} = \frac{1}{4}e^{3t}$$

$$x-3 = x\left(\frac{1}{4}e^{3t}\right)$$

$$x\left(1-\frac{1}{4}e^{3t}\right) = 3$$

$$\boxed{x = \frac{3}{1-\frac{1}{4}e^{3t}}}$$

check: $x_0 = \frac{3}{1-\frac{1}{4}} = \frac{3}{3/4} = \frac{4}{3} \cdot 3 = 4 \checkmark$

2d) Carefully explain why the behavior of your solution (2c) is consistent with your phase portrait in (2a). (3 points)

is consistent: $x_0 = 4 > 3 \implies$ doomsday.

if occurs when denominator of $x(t) \to 0$.

$$1 - \frac{1}{4}e^{3t} = 0$$

$$1 = \frac{1}{4}e^{3t}$$

$$4 = e^{3t}$$

$$\ln 4 = 3t$$

$$\boxed{t = \frac{\ln 4}{3}}$$

3) Consider the initial value problem

$$x''(t) + 6 \cdot x'(t) + 5 \cdot x = 0$$
$$x(0) = 0$$
$$x'(0) = 8$$

which could arise in a damped mass- spring problem.

3a) Find the general solution to the differential equation. What type of damping is exhibited here?
(10 points)

$$x = e^{rt}$$

$$p(r) = r^2 + 6r + 5$$
$$= (r+5)(r+1)$$

$$x(t) = c_1 e^{-5t} + c_2 e^{-t} \qquad \underline{overdamped}$$

3b) Solve the initial value problem.
(6 points)

$$x(0) = 0 = c_1 + c_2$$
$$x'(0) = 8 = -5c_1 - c_2$$

$$\left(x'(t) = -5c_1 e^{-5t} - c_2 e^{-t} \right)$$

$$\begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$$\boxed{x(t) = -2e^{-5t} + 2e^{-t}}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = +\frac{1}{4} \begin{bmatrix} -1 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -8 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

3c) If this was a mass spring system, we were using the metric system, and the mass was 2 kg, then what would be the damping coefficient and spring constant values? Include correct units.
(4 points)

$$mx'' + cx' + kx = 0$$

$$2(x'' + 6x' + 5x) = 0$$

$$2x'' + 12x' + 10x = 0$$

$$c = 12 \text{ kg/sec} \qquad k = 10 \text{ N/m}$$

4a) What is the dimension of the space of solutions $y(x)$ to the differential equation

$$y'''(x) + 3 \cdot y''(x) + 12 \cdot y'(x) + 10 \cdot y(x) = 0 \ ?$$

(3 points)

$$\boxed{n = 3}$$

4b) Find a basis for the solution space in 4a. (Hint: $r = -1$ is one root of the characteristic polynomial.)

(12 points)

$$p(r) = r^3 + 3r^2 + 12r + 10$$

$$p(-1) = -1 + 3 - 12 + 1 = 0 \checkmark$$

$$= (r+1)\left[(r+1+3i)(r+1-3i)\right]$$

$$r+1 \overline{\smash{\big)}\, r^3 + 3r^2 + 12r + 10} \quad \longrightarrow \quad = (r+1)^2 + 9$$

quotient: $r^2 + 2r + 10$

$$\underline{r^3 + r^2}$$
$$2r^2 + 12r$$
$$\underline{2r^2 + 2r}$$
$$10r + 10$$
$$\underline{10r + 10}$$
$$0$$

roots $r = -1, \ -1 \pm 3i$

$$\left\{ e^{-x}, \ e^{-x}\cos 3x, \ e^{-x}\sin 3x \right\}$$

4c) Prove the functions you found in your part (4b) basis are linearly independent. Hint: if you consider the linear independence equation you may notice a way to simplify it before using your usual approaches.

(5 points)

Let

$$c_1 e^{-x} + c_2 e^{-x}\cos 3x + c_3 e^{-x}\sin 3x \equiv 0$$

$$\Rightarrow \quad c_1 + c_2 \cos 3x + c_3 \sin 3x \equiv 0$$

$$D_x: \quad -3c_2 \sin 3x + 3c_3 \cos 3x \equiv 0$$

$$D_x^2: \quad -9c_2 \cos 3x - 9c_3 \sin 3x \equiv 0.$$

$$\begin{bmatrix} 1 & \cos 3x & \sin 3x \\ 0 & -3\sin 3x & 3\cos 3x \\ 0 & -9\cos 3x & -9\sin 3x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$$

@ $x = 0$: Wronskian matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & -9 & 0 \end{bmatrix}$; $\det = 27 \neq 0$ so $\vec{c} = \vec{0}$.

5a) Derive the formulas which express C,α in terms of A,B in the function identity

$$A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$$

(5 points)

(You don't need to worry about inverse trig functions - it suffices to write down expressions for $C, \cos(\alpha), \sin(\alpha)$, in terms of A, B.)

$$C \cos(\omega t - \alpha) = C(\cos \omega t \cos \alpha + \sin \omega t \sin \alpha)$$
$$= (C \cos \alpha) \cos \omega t + (C \sin \alpha) \sin \omega t$$

since $\cos \omega t, \sin \omega t$ lin. ind., must have

$$\boxed{C \cos \alpha = A, \quad C \sin \alpha = B}$$

So $A^2 + B^2 = C^2(\cos^2 \alpha + \sin^2 \alpha) = C^2$

So $\boxed{C = \sqrt{A^2 + B^2}}$

So also $\cos \alpha = \dfrac{A}{C}$, $\sin \alpha = \dfrac{B}{C}$

5b) Use Euler's formula and trigonometric identities to verifty that if $r = a + b \cdot i$, then $\dfrac{d}{dx} e^{r \cdot x} = r \cdot e^{r \cdot x}$.

(5 points)

$$e^{(a+ib)x} = e^{ax} e^{ibx}$$
$$= e^{ax}(\cos bx + i \sin bx)$$
$$= e^{ax} \cos bx + i(e^{ax} \sin bx)$$

thus $D_x e^{(a+bi)x} = e^{ax}(a \cos bx - b \sin bx) + i \, e^{ax}(a \sin bx + b \cos bx)$

$\uparrow$ same

on the other hand

$$r e^{rx} = (a + bi)(e^{ax})(\cos bx + i \sin bx)$$
$$= e^{ax}[(a \cos bx - b \sin bx) + i(b \cos bx + a \sin bx)]$$

same.

thus $D_x e^{rx} = r e^{rx}$!