Math 2280-1
Monday April 26
§ 9.5-9.6 heat and wave partial differential equations

9.5 1 space dimension heat equation for temperature $u(x,t)$ ← time $t \geq 0$

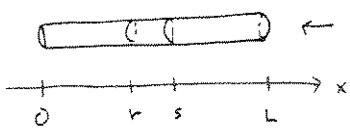
$\uparrow$ location $0 \leq x \leq L$

$$\boxed{u_t = k\,u_{xx}}$$

(or $u_t = k u_{xx} + f(x,t)$ for non-homogeneous heat eqtn)

"k" is called thermal diffusivity, depends on material, see table p. 622

derivation



← cross-sectional area a

"Heat" is a measurement of energy

If $u(x,t)$ is absolute temp (e.g. °Kelvin)

then the energy in test interval

$r \leq x \leq s$ is

$$E(t) = \int_r^s u(x,t)\, c\, \delta\, \underbrace{a\, dx}_{dV\ cm^3}$$

δ: density gm/cm^3; $\delta\, a\, dx$ = dm mass gm

c: specific heat calories/gm

= energy required to heat 1 gm by 1° C,

$c\,\delta\, a\, dx$ = energy required to heat dx-piece 1°.

$u(x,t)\, c\,\delta\, a\, dx$ = energy in dx-piece.

Assuming lateral insulation

(so heat can only escape through ends),

assume

$$\frac{dE}{dt} = K a \left(u_x(s) - u_x(r)\right) \qquad \text{Heat flux proportional to temperature gradient}$$

$$\overset{\parallel}{\int_r^s u_t\, c\delta a\, dx} = \overset{\parallel\ \text{FTC}}{K a \int_r^s u_{xx}\, dx}$$

$\lim_{s \to r} \frac{1}{s-r}$: $\Rightarrow u_t\, c\delta a = K a\, u_{xx}$

$$\boxed{u_t = k u_{xx}, \qquad k = \frac{K}{c\delta}}$$

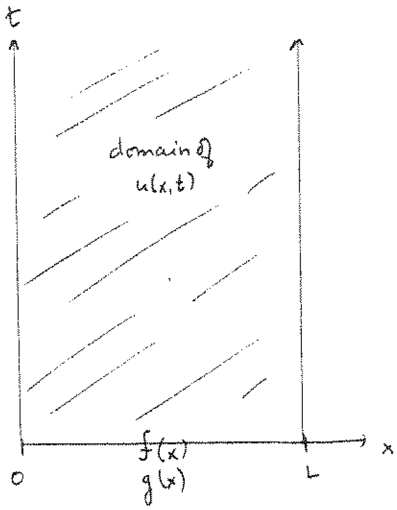
- If u satisfies $u_t = k u_{xx}$ then so does $v = c_1 u + c_2$, so you may use Celsius, Farenheit, Kelvin
- $L(u) = u_t - k u_{xx}$ is linear so principle of superposition holds.

Two of the most-usually studied IBVP's (Initial boundary value problems)

type 1

$$\begin{cases} u_t = k u_{xx} & 0<t<\infty,\ 0<x<L \\ u(x,0) = f(x) & \text{init. temp. } 0<x<L \\ u(0,t) = 0 & t>0 \\ u(L,t) = 0 & t>0 \end{cases}$$

boundary temp held const. (need not always be zero).



type 2

$$\begin{cases} u_t = k u_{xx} & 0<t<\infty,\ 0<x<L \\ u(x,0) = g(x) & 0<x<L \\ u_x(0,t) = 0 & t>0 \\ u_x(L,t) = 0 & t>0 \end{cases}$$

no heat flux thru boundary (insulated end condition)

Example 1 $\boxed{f(x) = \sin\left(\frac{\pi}{L}x\right)}$

product sol'n!

$$u(x,t) = v(t)\sin(\omega x), \quad v(0) = 1$$
$$u_t = v'(t)\sin\omega x$$
$$u_{xx} = v(t)(-\omega^2)\sin\omega x$$
$$\Leftrightarrow \begin{cases} v' = -k\omega^2 v \\ v(0) = 1 \end{cases} \Leftrightarrow v(t) = e^{-k\omega^2 t}$$

so $\boxed{u(x,t) = \sin\left(\frac{\pi}{L}x\right) e^{-k\left(\frac{\pi}{L}\right)^2 t}}$

solves 1! also $\boxed{u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{\pi}{nL}\right)^2 t}}$

general sol'n, use sine series for f & superpose!

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L}nx\right)$$
$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L}nx\right) e^{-k\left(\frac{\pi}{L}n\right)^2 t}$$

2 $g(x) = \cos\left(\frac{\pi}{L}x\right)$

$\boxed{u(x,t) = \cos\left(\frac{\pi}{L}x\right) e^{-k\left(\frac{\pi}{L}\right)^2 t}}$! (see LHS)

general sol'n, use cosine series for $g(x)$ & superpose product sol'ns!

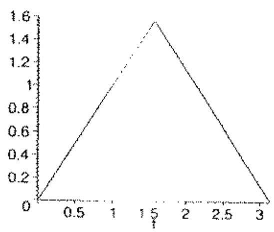
$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}nx\right)$$
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}nx\right) e^{-k\left(\frac{\pi}{L}n\right)^2 t}$$

(this method is called "separation of variables")

pictures next page, with tent function initial temperature, and $k=1$

(3)

```
> tent:=t->(1-Heaviside(t-Pi/2))*t +Heaviside(t-Pi/2)*(Pi-t):
> plot(tent(t),t=0..Pi,color=black);
```



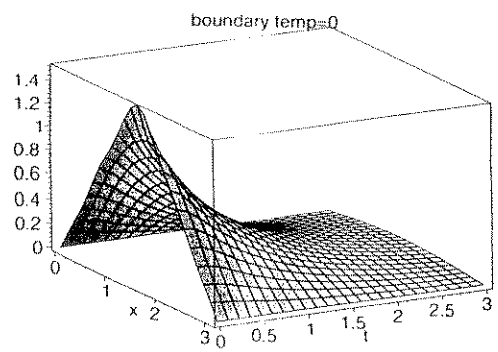
```
> sineseries:=proc(ff,L,b)    #ff=function, L=half period
  local m,  #dummy letter to index coefficients
        s;  #domain variable
  assume(m,integer):
  b:=m->simplify(2/L*int(ff(s)*sin(Pi/L*m*s),s=0..L)):
  end:
> sineseries(tent,Pi,B):
> B(n);
```

$$\frac{4\sin\left(\frac{\pi n\sim}{2}\right)}{\pi n\sim^2}$$

```
> u1:=(x,t)->sum(B(n)*sin(n*x)*exp(-n^2*t),n=1..10);
```

$$u1 := (x,t) \to \sum_{n=1}^{10} \mathrm{B}(n)\sin(n\,x)\,e^{(-n^2 t)} \qquad (k=1)$$

```
> plot3d(u1(x,t),x=0..Pi,t=0..3,axes=boxed,title='boundary temp=0');
```



$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \text{tent}(x) \\ u(0,t) = u(\pi,t) = 0 \quad t>0 \end{cases}$$

fixed boundary temp.

$\lim_{t\to\infty} u(x,t) = ?$

```
> cosseries:=proc(ff,L,a)    #ff=function, L=half period
  local m,  #dummy letter to index coefficients
        s;  #domain variable
  assume(m,integer):
  a:=m->simplify(2/L*int(ff(s)*cos(Pi/L*m*s),s=0..L)):
  end:
> cosseries(tent,Pi,A):
> A(0);
  A(n);
```

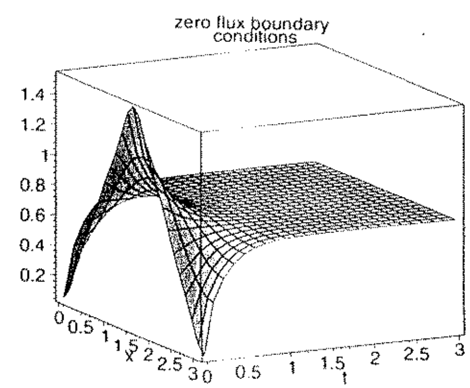
$$\frac{\pi}{2}$$

$$-\frac{2\left((-1)^{n\sim} - 2\cos\left(\frac{\pi n\sim}{2}\right) + 1\right)}{\pi n\sim^2}$$

```
> u2:=(x,t)->Pi/4+sum(A(n)*cos(n*x)*exp(-n^2*t),n=1..10);
```

$$u2 := (x,t) \to \frac{\pi}{4} + \left(\sum_{n=1}^{10} \mathrm{A}(n)\cos(n\,x)\,e^{(-n^2 t)}\right) \qquad (k=1)$$

```
> plot3d(u2(x,t),x=0..Pi,t=0..3,axes=boxed,title='zero flux boundary
  conditions');
```

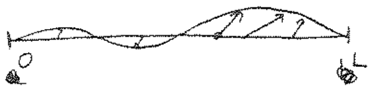


$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \text{tent}(x) \\ u_x(0,t) = u_x(\pi,t) = 0 \end{cases}$$

no heat flux

$\lim_{t\to\infty} u(x,t) = ?$

§9.6
Vibrating strings & the wave equation



vibrating string.

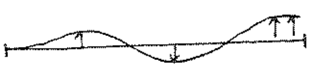
We assume tension in string changes as it is stretched,

i.e. $T = \mathbb{T}(\rho)$ $\quad \rho$ = density $\quad$ (probably $\mathbb{T}(\rho) \searrow$ fcn ρ)

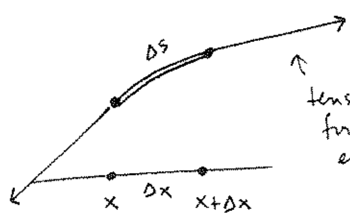
$T = \underbrace{\mathbb{T}(\rho_0)}_{T_0} + \mathbb{T}'(\rho_0)(\rho - \rho_0)$ $\quad$ linearized model; $\quad T_0, \rho_0$ for stationary (but stretched) string.

<u>case 1</u> transverse (vertical) vibrations

$y = y(x,t)$ = vertical displacement



isolate a tiny piece, apply Newton's Law, linearize, to deduce wave eqtn:



$$\underbrace{\rho \Delta s}_{\text{mass}} \underbrace{y_{tt}}_{\text{accel}} = \overbrace{\overset{\text{tension}}{T} \frac{y_x}{\sqrt{1+y_x^2}} \Bigg]_x^{x+\Delta x}}^{\text{net forces}}$$

vertical component of unit tang. vector to profile curve

$$\begin{bmatrix} 1 \\ y_x \end{bmatrix} \frac{1}{\sqrt{1+y_x^2}}$$

linearize

$$\frac{\Delta s}{\Delta x} \approx \sqrt{1+y_x^2} \approx 1$$

(y_x small $\Rightarrow y_x^2$ negligable)

$$\rho_0 (\Delta x) y_{tt} \approx T_0 \left(y_x(x+\Delta x) - y_x(x) \right)$$

$\rho \Delta s = \rho_0 \Delta x$

$\Rightarrow \rho = \rho_0 \frac{ds}{dx} = \rho_0 \sqrt{1+y_x^2} \approx \rho_0$

$\Rightarrow T \approx T_0$

$$\rho_0 y_{tt} \approx T_0 \left(\frac{y_x(x+\Delta x) - y_x(x)}{\Delta x} \right)$$

$$y_{tt} = \left(\frac{T_0}{\rho_0}\right) y_{xx}$$

$$\boxed{y_{tt} = a^2 y_{xx}}$$

$a = \sqrt{\frac{T_0}{\rho_0}}$ turns out to be wave speed.

<u>case 2</u> longitudinal motion, i.e. in direction of string,

you get

$$x_{tt} = b^2 y_{xx}$$

where $b^2 = -T'(\rho_0)$,

different speed!

What does "a" mean?

Special solution:

Let $f(z)$ a function of 1-variable.

consider $y(x,t) = f(x-at)$

$$y_t = f'(x-at)(-a) \qquad \text{chain rule!}$$

$$y_{tt} = f''(x-at)\, a^2$$

$$y_x = f'(x-at)\, 1$$

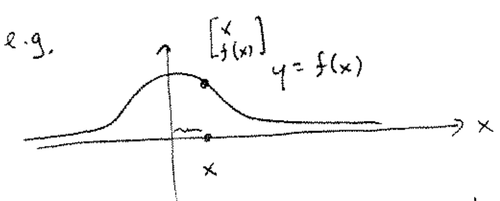
$$y_{xx} = f''(x-at)\, 1$$

$$y_{tt} = a^2 y_{xx} \quad !$$

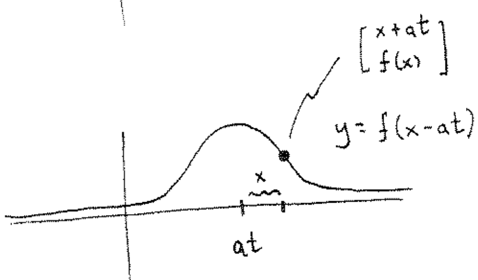
(also, $y(x,t) = f(x+at)$ solves the wave equation.)

$y(x,t) = f(x-at)$ is a wave (with constant "profile") moving to the right $\boxed{\text{with speed } a\,!}$

e.g.



profile at $t=0$



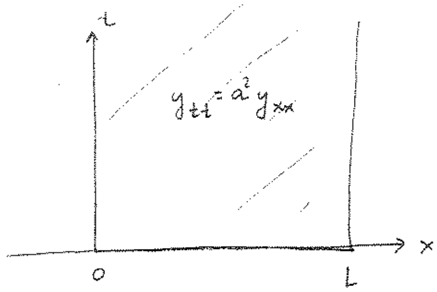
profile at time t has moved over by amount at

$y(x,t) = f(x+at)$ is a wave moving to the left!

(It will turn out that every solution to the wave eqtn $y_{tt} - a^2 y_{xx} = 0$ is a superposition of waves traveling to the left or right with speed a!)

In case 1, $a = \sqrt{\frac{T_0}{\rho_0}}$, so higher tension $\Rightarrow$ faster speed
lower density $\Rightarrow$ faster speed.

Natural IBVP's for wave eqtn:



$$\begin{cases} y_{tt} = a^2 y_{xx} & t>0,\ 0<x<L \\ y(x,0) = f(x) & \text{initial displacement} \\ y_t(x,0) = g(x) & \text{initial velocity} \end{cases}$$

plus ① $y(0,t) = y(L,t) = 0$ fixed endpoints **type 1**

or ② $y_x(0,t) = y_x(L,t) = 0$ free endpoints **type 2**

Use Fourier series & superposition: $(2 \times 2 = 4)$, and you can solve type 1 & type 2 problems!

① $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0) = f(x) \quad (y_t(x,0) = 0)$
$y(0,t) = y(L,t) \equiv 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0) = g(x) \quad (y(x,0) = 0)$
$y(0,t) = y(L,t) = 0$

② $\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0) = f(x) \quad (y_t(x,0) = 0)$
$y_x(0,t) = y_x(L,t) = 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0) = g(x) \quad (y(x,0) = 0)$
$y_x(0,t) = y_x(L,t) = 0$

It's possible I'll have a slinky and we can demo this!