Math 2280-1
Friday April 23

§9.4 : forced oscillations via Fourier
(and 9.3 : even & odd extensions)

HW for Friday April 30, 5:00 pm
(classes actually end Wednesday).

9.3 1, 9, 17, 20
9.4 1, 7, 9, 13
9.5 1, 2, 7
9.6 1, 5

- At the end of Wednesday we checked how to derive Fourier series for piecewise continuous, 2L-periodic functions

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

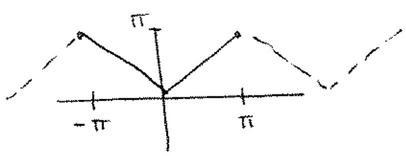
$$\langle f, g\rangle = \frac{1}{L}\int_{-L}^{L} f(t) g(t)\, dt$$

$$a_0 = \langle f, 1\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\, dt$$

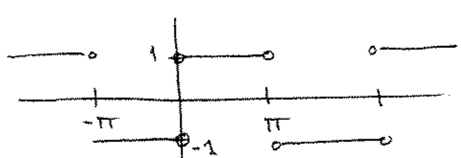
$$a_n = \langle f, \cos \tfrac{\pi}{L} n t\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{\pi}{L} n t\right) dt$$

$$b_n = \langle f, \sin \tfrac{\pi}{L} n t\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{\pi}{L} n t\right) dt$$

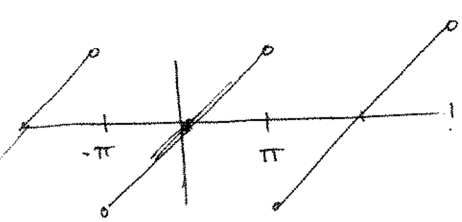
- We have computed Fourier series:



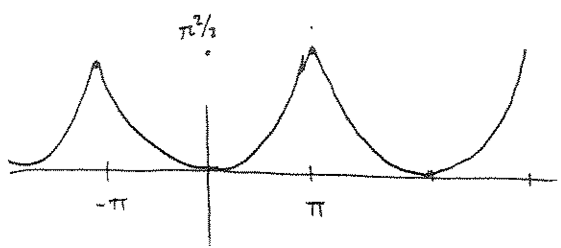
$$\text{tent}(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$



$$\text{square}(t) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n} \qquad \left(= \frac{d}{dt}\,\text{tent}(t)\right)$$



$$\text{sawtooth}(t) \sim 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$



$$\text{parabola}(t) = \int_0^t \text{sawtooth}(\tau)\, d\tau \qquad \text{for } -\pi \le t \le \pi$$

$$\left(\frac{t^2}{2}\right) \qquad = 2\sum_{n=1}^{\infty} (-1)^n \frac{\cos nt}{n^2} + \frac{\pi^2}{6}$$

error in Wed notes.

- To get Fourier series linear combinations of fcns, take linear combos of Fourier series because Fourier coeff's are linear wrt the function input.

- We've noticed that

  <u>odd</u> 2L-periodic functions have Fourier series with all $a_n = 0$,
  i.e. they are <u>sine</u> series

  <u>even</u> 2L-periodic functions have Fourier series with all $b_n = 0$
  i.e. they are <u>cosine</u> series

So, if $f : [0, L] \to \mathbb{R}$ is p.w. continuous
you can extend to $f : [-L, L] \to \mathbb{R}$
as even: $f(-t) := f(t) \quad 0 \le t \le L$
and then to $f : \mathbb{R} \to \mathbb{R}$ as 2L-periodic $\left(f(t + 2Ln) := f(t) \ \forall n \in \mathbb{Z} \right)$

$$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right)$$ <u>cosine series for f</u>

OR
you can extend
to $f : [-L, L] \to \mathbb{R}$
as odd: $f(-t) := f(t) \quad 0 \le t \le L$
and then to $f : \mathbb{R} \to \mathbb{R}$ as 2L-periodic

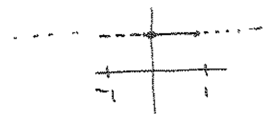
$$\Rightarrow f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$ <u>sine series for f</u>

(also page 2.)
Wed.

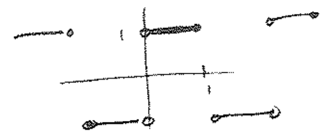
examples (we can steal)

① Let $f(t) = 1$, $0 \le t \le 1$

What is the cosine series for f?



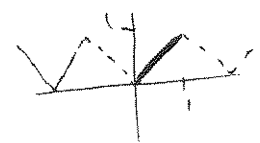
What is the sine series for f?



Hint: Rescale square(t) from page 1.

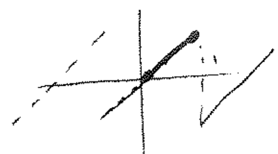
② Let $g(t) = t$, $0 \le t \le 1$

What is the cosine series for g?



Hint: page 1.

What is the sine series for g?



Fourier & springs (59.4)

Let $f(t)$ be $2L$-periodic,
consider

$$x''(t) + \omega_0^2 x = f(t)$$

$x_H = A\cos\omega_0 t + B\sin\omega_0 t$.

so $T_0 =$ "natural period" $= \frac{2\pi}{\omega_0}$.

But when we played "guess the resonance" we saw that even if f's period $2L \neq T_0$ you may get resonance. (Conversely, even if f has period $\frac{2\pi}{\omega_0}$ you might not get resonance).

Here's the answer (for whether or not resonance happens)

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\tfrac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{\pi}{L} n t\right)$$

Use (infinite) superposition!

$$x'' + \omega_0^2 x = 1 \quad \longrightarrow \quad x_p = \frac{1}{\omega_0^2}$$

$$x'' + \omega_0^2 x = \begin{cases} \cos\omega t & \begin{cases} x_p = \frac{1}{\omega_0^2 - \omega^2}\cos\omega t & \omega \neq \omega_0 \\ x_p = \frac{1}{2}\frac{t}{\omega_0}\sin\omega_0 t & \omega = \omega_0 \end{cases} \\ \sin\omega t & \begin{cases} x_p = \frac{1}{\omega_0^2 - \omega^2}\sin\omega t & \omega \neq \omega_0 \\ x_p = -\frac{t}{2\omega_0}\cos\omega_0 t & \omega = \omega_0 \end{cases} \end{cases}$$

So, soln to $x''(t) + \omega_0^2 x(t) = f(t)$

is

$$x(t) = \frac{a_0}{\omega_0^2} + \sum_{n=1}^{\infty} a_n \frac{1}{\omega_0^2 - \left(\frac{\pi}{L}n\right)^2}\cos\left(\tfrac{\pi}{L}nt\right) + \sum_{n=1}^{\infty} b_n \frac{1}{\left(\omega_0^2 - \left(\frac{\pi}{L}n\right)^2\right)}\sin\left(\tfrac{\pi}{L}nt\right) + \underbrace{x_H(t)}_{c_1\cos\omega_0 t + c_2\sin\omega_0 t}$$

provided $\nexists\, n$ s.t. $\frac{\pi}{L}n = \omega_0$ i.e. $\frac{2\pi}{\omega_0} = \frac{2L}{n} = \frac{\text{forcing period}}{n}$

If the natural period is an integer fraction $\frac{1}{n}\cdot$ forcing period, then there is resonance iff one of the corresponding a_n, or $b_n \neq 0$!!

i.e. the forcing period is a multiple of the natural period.

yields term $a_n \frac{t}{2\omega_0}\sin\omega_0 t$ or $b_n\left(\frac{-t}{2\omega_0}\cos\omega_0 t\right)$

SEE MAPLE!