Math 2280-1

Wednesday April 21

§9.3-9.4

Finish Tuesday notes:

- f' Fourier series by differentiating f's
- If f is a finite linear combo of 1, $\cos nt$'s, & $\sin nt$'s then that's its Fourier series } pages 1-2
- 2L periodic fcn Fourier series } page 3

§9.3

We've noticed that

<u>odd</u> 2L-periodic fcns' Fourier series are <u>sine series</u>, because all $a_n = 0$.

<u>even</u> 2L-periodic fcns' Fourier series are <u>cosine series</u> because all $b_n = 0$.

So, If $f : [0, L] \to \mathbb{R}$

- you can extend $f : [-L, L] \to \mathbb{R}$ as an even fcn $f(t) := f(-t)$, and then to $\mathbb{R}$ as a 2L-periodic fcn $\left(f(t+2kL) := f(t) \right)_{-L<t<L}$

$$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right)$$

<u>cosine series for f</u>

Or,

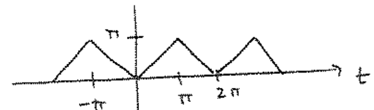
- you can extend $f : [-L, L] \to \mathbb{R}$ as an odd fcn $f(-t) := f(t) \quad 0<t<L$, and then to $\mathbb{R}$ as a 2L-periodic fcn

$$\Rightarrow f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

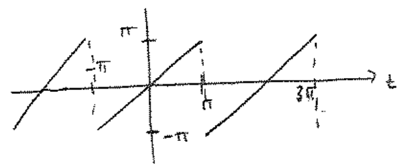
<u>sine series for f.</u>

(on the interval $0<t<L$ either series will converge to $f(t)$, wherever f is differentiable!)

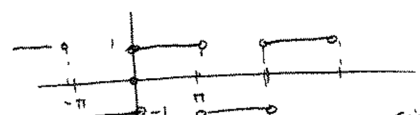
(Useful in §9.5-9.6)



$$\text{tent}(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$



$$\text{sawtooth}(t) \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$



$$\text{square}(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$$

If f is 2L-periodic

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(t) g(t)\, dt$$

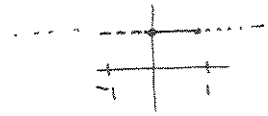
$$a_0 = \langle f, 1 \rangle = \frac{1}{L} \int_{-L}^{L} f(t)\, dt$$

$$a_n = \langle f, \cos \tfrac{\pi}{L} n t \rangle = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{\pi}{L} n t\right) dt$$

$$b_n = \langle f, \sin \tfrac{\pi}{L} n t \rangle = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{\pi}{L} n t\right) dt$$

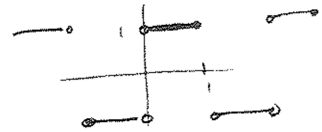
examples (we can steal)

1. Let $f(t) = 1, \quad 0 \le t \le 1$



What is the cosine series for f?

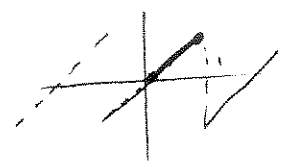
What is the sine series for f?



2. Let $g(t) = t, \quad 0 \le t \le 1$

What is the cosine series for g?

What is the sine series for g?



We talked about differentiating Fourier series term by term.
There is also:

Theorem: If f is p.w. cont., $2L$-periodic, with Fourier series (page 605)

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

integrated term by term.

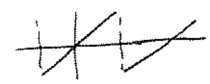
then the antiderivative

$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi}{L}t\right) + b_n\left(\frac{L}{n\pi}\right)\left[-\cos\left(\frac{n\pi}{L}t\right)+1\right]$$

series on the right converges to this antideriv!

Example (9.3 #19)

start with $t = \text{sawtooth}(t) \sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$ $\quad -\pi < t < \pi$



Integrate once to deduce

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty}(-1)^n\frac{\cos nt}{n^2} + \frac{\pi^2}{12}$$

Hint: Use $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n \text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$

(In HW need to get all the way to $\frac{t^4}{24}$!)

Fourier & springs (59.4)

Let $f(t)$ be $2L$-periodic, consider

$$x''(t) + \omega_0^2 x = f(t)$$

$x_H = A\cos\omega_0 t + B\sin\omega_0 t.$

so $T_0 =$ "natural period" $= \frac{2\pi}{\omega_0}$.

But when we played "guess the resonance" we saw that even if f's period $2L \neq T_0$ you may get resonance. (Conversely, even if f has period $\frac{2\pi}{\omega_0}$ you might not get resonance).

Here's the answer (for whether or not resonance happens)

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\tfrac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{\pi}{L} n t\right)$$

Use (infinite) superposition!

$$x'' + \omega_0^2 x = 1 \longrightarrow x_p = \frac{1}{\omega_0^2}$$

$$x'' + \omega_0^2 x = \begin{cases} \cos\omega t & \begin{cases} x_p = \frac{1}{\omega_0^2 - \omega^2}\cos\omega t & \omega \neq \omega_0 \\ x_p = \frac{1}{2}\frac{t}{\omega_0}\sin\omega_0 t & \omega = \omega_0 \end{cases} \\ \sin\omega t & \begin{cases} x_p = \frac{1}{\omega_0^2 - \omega^2}\sin\omega t & \omega \neq \omega_0 \\ x_p = -\frac{t}{2\omega_0}\cos\omega_0 t & \omega = \omega_0 \end{cases} \end{cases}$$

So, soln to $x''(t) + \omega_0^2 x(t) = f(t)$

is

$$x(t) = \frac{a_0}{\omega_0^2} + \sum_{n=1}^{\infty} a_n \frac{1}{\omega_0^2 - (\frac{\pi}{L}n)^2}\cos\left(\tfrac{\pi}{L}nt\right) + \sum_{n=1}^{\infty} b_n \frac{1}{(\omega_0^2 - (\frac{\pi}{L}n)^2)}\sin\left(\tfrac{\pi}{L}nt\right) + \underbrace{x_H(t)}_{c_1\cos\omega_0 t + c_2\sin\omega_0 t}$$

provided $\nexists\, n$ s.t. $\frac{\pi}{L}n = \omega_0$ i.e. $\frac{2\pi}{\omega_0} = \frac{2L}{n} = \frac{\text{forcing period}}{n}$

If the natural period is an integer fraction $\frac{1}{n}\cdot$ forcing period, then there is resonance iff one of the corresponding a_n, or $b_n \neq 0$!!

i.e. the forcing period is a multiple of the natural period.

yields term $a_n \frac{t}{2\omega_0}\sin\omega_0 t$ or $b_n\left(\frac{-t}{2\omega_0}\cos\omega_0 t\right)$