

Math 2280-1
Monday April 19

Finish convolution theorem
and playing the resonance game,
Friday notes. Then...

①

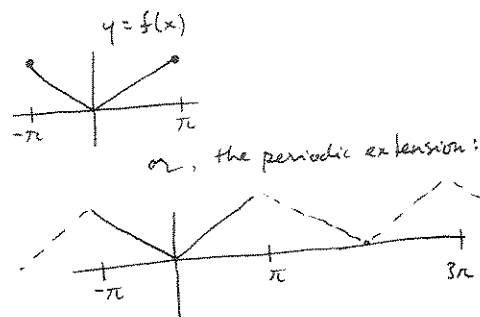
Chapter 9: Fourier series & applications to DE's and PDE's

↑
partial differential equations

Recall (?) from 2270:

Let $f: [-\pi, \pi] \rightarrow \mathbb{R}$ be piecewise continuous

(or equivalently, extend $f: \mathbb{R} \rightarrow \mathbb{R}$
as a 2π -periodic, p.w. cont. function)



Fourier coefficients of f :

$$a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt \quad n \in \mathbb{N}$$

$$b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt \quad n \in \mathbb{N}$$

$$\text{Fourier series for } f: \quad f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

Motivation (2270 "orthogonality" chapter)

$V := \{ f: \mathbb{R} \rightarrow \mathbb{R}, f \text{ is } 2\pi \text{ periodic } (f(x+2\pi) = f(x) \forall x), \text{ and p.w. continuous} \}$

inner product

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) dt \quad \text{satisfies the axioms}$$

$$\begin{cases} a) \langle f, f \rangle \geq 0, \langle f, f \rangle = 0 \text{ iff } f=0 \\ b) \langle f, g \rangle = \langle g, f \rangle \\ c) \langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle \\ d) \langle sf, g \rangle = \langle f, sg \rangle = s \langle f, g \rangle \end{cases} \quad \forall f, g \in V, s \in \mathbb{R}$$

So, $\|f\| = \langle f, f \rangle^{1/2}$, $f \perp g$ iff $\langle f, g \rangle = 0$, $\angle f, g = \cos^{-1} \left(\frac{\langle f, g \rangle}{\|f\| \|g\|} \right)$, projection, etc!

$$\text{dist}(f, g) = \|f - g\|,$$

In particular, if $\{u_1, u_2, \dots, u_k\}$ are orthonormal and span a subspace W of V

then

$$\text{proj}_W f = \langle f, u_1 \rangle u_1 + \langle f, u_2 \rangle u_2 + \dots + \langle f, u_k \rangle u_k \dots$$

is the nearest function in W , to f .

(just like you proved for dot product, using only (a)-(d)!))

Theorem Let $V_N = \text{span} \left\{ \frac{1}{\sqrt{2}}, \cos t, \cos 2t, \dots, \cos Nt, \sin t, \sin 2t, \dots, \sin Nt \right\}$

The $2N+1$ functions in this list are orthonormal! (With respect to page 1 inner product)

Thus for any $f \in V$,

$$\text{proj}_{V_N} f = \underbrace{\langle f, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}}}_{\frac{a_0}{2}} + \sum_{n=1}^N \underbrace{\langle f, \cos nt \rangle}_{a_n} \cos nt + \sum_{n=1}^N \underbrace{\langle f, \sin nt \rangle}_{b_n} \sin nt$$

i.e., the truncated Fourier series

(the idea is that as $N \rightarrow \infty$, the $\text{proj}_{V_N} f$ approximate f effectively; see later theorems)

proof:

Check that $\left\{ \frac{1}{\sqrt{2}}, \cos t, \cos 2t, \dots, \cos Nt, \sin t, \sin 2t, \dots, \sin Nt \right\}$ are orthonormal:

Hint:

$$\cos(m+k)t = \cos mt \cos kt - \sin mt \sin kt$$

$$\sin(m+k)t = \cos mt \sin kt + \sin mt \cos kt$$

$\Rightarrow$

$$\cos mt \cos kt = \frac{1}{2} [\cos(m+k)t + \cos(m-k)t] \quad (\text{even if } m=k!)$$

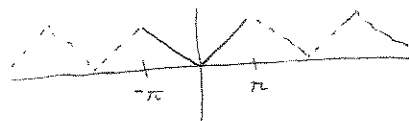
$$\sin mt \sin kt = \frac{1}{2} [\cos(m-k)t - \cos(m+k)t] \quad (\text{even if } m=k!)$$

$$\cos mt \sin kt = \frac{1}{2} [\sin(m+k)t + \sin(-m+k)t]$$

Example 1 : an even function ($f(-x) = f(x)$)

$$f(t) = |t| \text{ for } -\pi \leq t \leq \pi$$

extended to be 2π -periodic :



find a_0, a_n, b_n

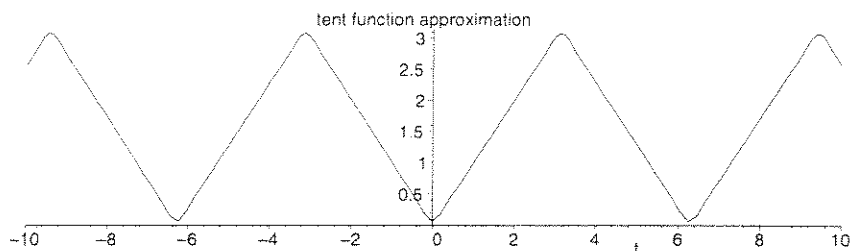
ans :

$$f \sim \frac{\pi}{2} - \frac{4}{\pi} \left(\cos t + \frac{1}{9} \cos 3t + \frac{1}{25} \cos 5t + \dots \right)$$

```
> f9:=t->Pi/2-4/Pi*sum(cos((2*j+1)*t)/(2*j+1)^2,j=0..4);
```

$$f_9 := t \rightarrow \frac{\pi}{2} - \frac{4 \left(\sum_{j=0}^4 \frac{\cos((2j+1)t)}{(2j+1)^2} \right)}{\pi}$$

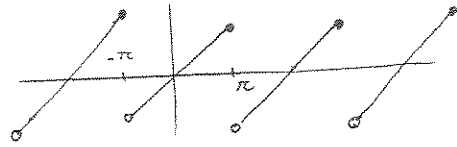
```
> plot1:=plot(f9(t),t=-10..10,color=black);
display({plot1},title='tent function approximation');
```



Example 2 an odd fcn ($f(-x) = -f(x)$)

$$f(t) = t \text{ for } -\pi < t \leq \pi$$

extended to be 2π -periodic



find a_0, a_n, b_n

Can you deduce anything
about Fourier coeffs of
 2π -periodic even functions?
of 2π -periodic odd functions?

ans:

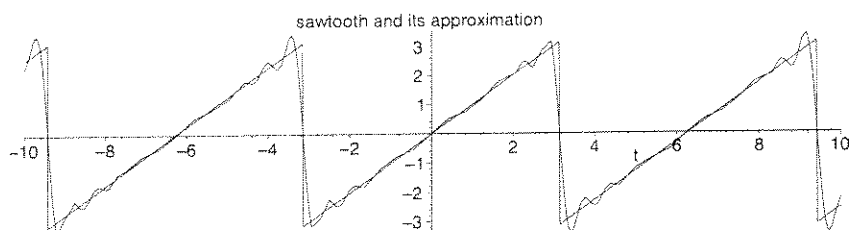
$$f \sim 2 \left(\sin t - \frac{\sin 3t}{3} + \frac{\sin 5t}{5} - \dots \right)$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$

```
> g10:=t->2*sum((-1)^(n+1)*sin(n*t)/n,n=1..10);
```

$$g10 := t \rightarrow 2 \left(\sum_{n=1}^{10} \frac{(-1)^{n+1} \sin(nt)}{n} \right)$$

```
> plot3:=plot(g10(t),t=-10..10,color=black):
plot4:=plot(t-2*Pi*Heaviside(t-Pi)-2*Pi*Heaviside(t-3*Pi)-
2*Pi*(Heaviside(t+Pi)-1)-2*Pi*(Heaviside(t+3*Pi)-1),t=-10..10,colo
r=black):
display({plot3,plot4},title='sawtooth and its approximation');
```



Convergence theorems: (Need more analysis than you know yet to prove these)

① Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and piecewise continuous.

$$\text{Let } f_N = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nt + \sum_{n=1}^N b_n \sin nt$$

$$\text{Then } \lim_{N \rightarrow \infty} \|f - f_N\| = \lim_{N \rightarrow \infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t) - f_N(t))^2 dt \right]^{1/2} = 0$$

↓
the way we
measure "distance" between
functions for our given inner product

② If f is as above and piecewise differentiable, with jump discontinuities, then for any

$$t_0 \text{ s.t. } f \text{ is differentiable at } t_0, \quad \lim_{N \rightarrow \infty} f_N(t_0) = f(t_0)$$

for any

$$t_0 \text{ s.t. } f \text{ has a jump at } t_0, \text{ from } f_-(t_0) := \lim_{t \nearrow t_0} f(t)$$

$$f_+(t_0) = \lim_{t \searrow t_0} f(t)$$

$$\text{then } \lim_{N \rightarrow \infty} f_N(t_0) = \frac{1}{2} (f_-(t_0) + f_+(t_0)), \text{ i.e. the average of the left \& right limits}$$

[you can see this illustrated in the sawtooth example].

③ If f is continuous, and f' is piecewise continuous, and if f is piecewise diffble

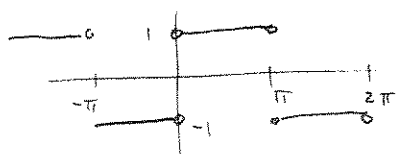
$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$\text{then } f' \sim \sum_{n=1}^{\infty} -n a_n \sin nt + \sum_{n=1}^{\infty} n b_n \cos nt$$

i.e. you can differentiate (& antidiff) term by term.

Example 3. Use Example 1 to find the square-wave Fourier series

$$sq(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0. \end{cases}$$



$$\text{ans: } sq \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$$