Math 2280-1
Wed April 14
§7.1-7.3 More magic Laplace transform

Fill in more of the table
do more examples.

	$f(t)$	$F(s)$	for $t \geq 0$, $\lvert f(t)\rvert \leq Ce^{Mt}$
	$f(t)$	$\int_0^\infty e^{-st} f(t)\,dt$	$s > M$
	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	$\mathcal{L}$ is linear!
	1	$1/s$	$s > 0$
	e^{at}	$1/(s-a)$	$s > Re(a)$
	$\cos kt$	$s/(s^2+k^2)$	
	$\sin kt$	$k/(s^2+k^2)$	
	$e^{at}\cos kt$	$(s-a)/((s-a)^2+k^2)$	
	$e^{at}\sin kt$	$k/((s-a)^2+k^2)$	
1a	$f'(t)$	$s F(s) - f(0)$	
1a	$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$	
1a	$f'''(t)$	$s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$ etc.	
1a	$\int_0^t f(\tau)\,d\tau$	$F(s)/s$	
1b	$t f(t)$	$-F'(s)$	
1b	$t^2 f(t)$	$F''(s)$	
1b	$t^3 f(t)$	$-F'''(s)$	
1b	$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
2	$e^{at} f(t)$	$F(s-a)$	
	1	$1/s$	
	t	$1/s^2$	
	t^n	$n!/s^{n+1}$	
	$t e^{at}$	$1/(s-a)^2$	
	$t^n e^{at}$	$n!/(s-a)^{n+1}$	
	$t \cos kt$	$(s^2-k^2)/(s^2+k^2)^2$	3 resonance!
	$\frac{t}{2k}\sin kt$	$s/(s^2+k^2)^2$	3 resonance!
	$\frac{1}{2k^3}[\sin kt - kt\cos kt]$	$1/(s^2+k^2)^2$	3 resonance!
	$(f*g)(t) = \int_0^t f(\tau) g(t-\tau)\,d\tau$	$F(s)G(s)$	4 how to $\mathcal{L}^{-1}$ products!

(this is in §7.4)

1b (numbered with 1a because derivatives on one side of the column corresponds to multiplication on the other side!

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\int_0^\infty e^{-(s+\Delta s)t} f(t)\,dt - \int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \lim_{\Delta s \to 0} \int_0^\infty f(t)\left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right] dt$$

$$\Big\downarrow {\scriptstyle \Delta s \to 0} \qquad \frac{d}{ds} e^{-st} = -t e^{-st}$$

(3210 to interchange limit & integral)

$$-\int_0^\infty e^{-st} \left(t f(t)\right) dt = -\mathcal{L}\{t f(t)\}(s)$$

then $\mathcal{L}\{t \cdot t f(t)\}(s) = -\frac{d}{ds} \underbrace{\mathcal{L}\{t f(t)\}(s)}_{-\frac{d}{ds}F(s)} = F''(s)$ etc.

Also $\frac{d}{ds} \underbrace{\int_s^\infty F(\sigma)\,d\sigma}_{G(s)} = -F(s)$

$$-G'(s) = F(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$t g(t) = f(t)$$

Table entries from 1b:

$\mathcal{L}\{1\} = 1/s$ (did)

$\Rightarrow \mathcal{L}\{t\cdot 1\} = -D_s\left(\frac{1}{s}\right) = \frac{1}{s^2} = s^{-2}$

$\Rightarrow \mathcal{L}\{t\cdot t\}(s) = -D_s\left(s^{-2}\right) = 2s^{-3}$

$\Rightarrow \mathcal{L}\{t\cdot t^2\}(s) = -D_s\left(2s^{-3}\right) = 3!\, s^{-4}$

$\Rightarrow$ (by induction) $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$

also, 3 follows:

$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$

$$\Rightarrow \mathcal{L}\{t\cos kt\}(s) = -D_s\left(\frac{s}{s^2+k^2}\right) = -\frac{(s^2+k^2)-s(2s)}{(s^2+k^2)^2} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$

$$\Rightarrow \mathcal{L}\{t\sin kt\}(s) = -k(-1)(s^2+k^2)^{-2}\,2s = 2k\left(\frac{s}{(s^2+k^2)^2}\right)$$

so $\mathcal{L}\left\{\frac{1}{2k}\, t\sin kt\right\}(s) = \frac{s}{(s^2+k^2)^2}$

Finally, (using what we know)

$$\left.\begin{aligned} \mathcal{L}\{t\cos kt\}(s) &= \frac{s^2-k^2}{(s^2+k^2)^2} \\ \mathcal{L}\left\{\tfrac{1}{k}\sin kt\right\}(s) &= \frac{s^2+k^2}{(s^2+k^2)^2} \end{aligned}\right\} \mathcal{L}\left\{\frac{1}{k}\sin kt - t\cos kt\right\}(s) = 2k^2\left(\frac{1}{(s^2+k^2)^2}\right)$$

so $\mathcal{L}\left\{\frac{1}{2k^3}\left[\sin kt - kt\cos kt\right]\right\}(s) = \frac{1}{(s^2+k^2)^2}$ $\blacksquare$

2 We already saw this for $\mathcal{L}\{e^{at}\cos kt\}(s) = \frac{s-a}{(s-a)^2+k^2}$

$\mathcal{L}\{e^{at}\sin kt\}(s) = \frac{k}{(s-a)^2+k^2}$

In general, $\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-st}e^{at}f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a) \quad \blacksquare$$

Table entries from 2:

$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}$

This just leaves 4, the convolution theorem.

First, applications – of what we've done so far!

example 1 $\begin{cases} x'' + 4x = 3\sin 2t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$

Resonance! (these used to take a really long time)

$$s^2 X(s) - 2s - 1 + 4X(s) = 3 \frac{2}{s^2+4}$$

$$X(s)[s^2+4] = \frac{6}{s^2+4} + 2s + 1$$

$$X(s) = \frac{6}{(s^2+4)^2} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$x(t) = 6 \frac{1}{2\cdot 8}[\sin 2t - 2t\cos 2t] + 2\cos 2t + \frac{1}{2}\sin 2t$$

$$= -\frac{3}{4} t\cos 2t + \frac{7}{8}\sin 2t + 2\cos 2t$$

example 2 §7.3 p. 466 (unforced damped spring)

$\begin{cases} x'' + 6x' + 34x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$

$\mathcal{L}$: $s^2 X(s) - 3s - 1 + 6(sX(s) - 3) + 34X(s) = 0$

$$X(s)(s^2+6s+34) = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$

$$= \frac{3(s+3) + 10}{(s+3)^2 + 25}$$

← complete the linear (?)

← complete the square

$$= 3 \frac{(s+3)}{(s+3)^2+25} + 2 \cdot \frac{5}{(s+3)^2+25}$$

3 F(s+3) for f(t) = cos 5t

G(s+3) for g(t) = sin 5t

$\mathcal{L}^{-1}$:

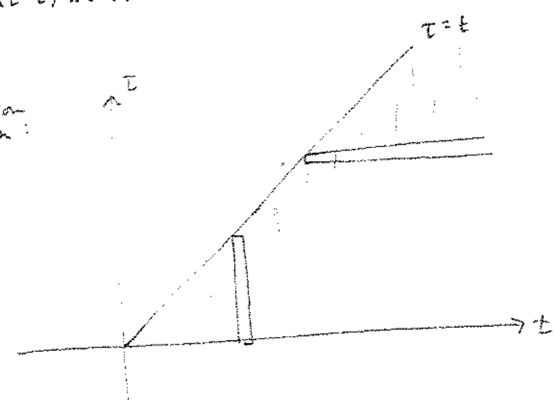
$$x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$$

proof of convolution theorem:
(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right] d\tau$$

$\tilde{t} = t - \tau$
$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}}\, g(\tilde{t})\,d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$

__example:__ verify the theorem
for $f(t) = \sin t$
$g(t) = \cos t$
(you may need trig id
$\sin^2\tau = \frac{1-\cos 2\tau}{2}$)

__Remark:__
$$f*g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$
$$= g*f(t)$$
(by substituting $\tilde{\tau} = t-\tau$)