Math 2280-1
Tuesday April 13

# Chapter 7 : Laplace transform : back to linear land.

a way of transforming IVP's for linear DE's, and linear systems of DE's, directly into solvable <u>algebra</u> problems.

let $f(t)$ be a function defined for $t \geq 0$ s.t. $f$ grows at most exponentially fast ( $|f(t)| \leq Ce^{Mt}$ some $C, M$)
Then the Laplace transform of $f$, $F(s) = \mathcal{L}\{f(t)\}(s)$ is defined for $s > M$ by

$$\underset{F(s)}{\underset{\|}{\mathcal{L}\{f(t)\}(s)}} := \int_0^\infty e^{-st} f(t)\, dt$$

notice that if $|f(t)| \leq Ce^{Mt}$ then $|e^{-st} f(t)| \leq Ce^{(M-s)t}$ so this improper integral converges for $s > M$.

<u>examples</u>

- $\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st} 1\, dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$ $(s > 0)$
- $\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt = \left.\frac{1}{a-s} e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a}$ $(s > a)$ (or if $a$ is complex, $s >$ Real part of $a$)
- Laplace transform is linear!

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s)$$
$$= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\, dt$$
$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t) dt$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

- $\mathcal{L}\{e^{ikt}\}(s) = \frac{1}{s-ik}$ see above!

// Euler

$$= \frac{s+ik}{s^2+k^2} \quad (\text{mult by } \tfrac{s+ik}{s+ik})$$

$$\mathcal{L}\{\cos kt + i \sin kt\}(s) = \frac{s}{s^2+k^2} + i\frac{k}{s^2+k^2}$$

$$\| $$

$$\mathcal{L}\{\cos kt\}(s) + i\,\mathcal{L}\{\sin kt\}(s)$$

Thus (equate real parts; equate imag parts)

- $\begin{cases} \mathcal{L}\{\cos kt\}(s) = \dfrac{s}{s^2+k^2} \\ \mathcal{L}\{\sin kt\}(s) = \dfrac{k}{s^2+k^2} \end{cases}$

easier than computing $\int_0^\infty e^{-st} \begin{matrix}\cos kt \\ \sin kt\end{matrix}\, dt$ directly!!

Exercise : What is $\mathcal{L}\{7-5e^{2t}+4\cos 2t\}(s)$

Laplace transform and DE's :

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{v}\,\underbrace{f'(t)dt}_{du} = e^{-st}f(t)\Big]_{t=0}^{\infty} - \int_0^\infty -se^{-st}f(t)\,dt$$

$dv = -se^{-st}dt \quad u = f(t)$

$$= -f(0) + sF(s) \qquad s > M$$

exp growth rate of f

so
$$\mathcal{L}\{f''(t)\}(s) = -f'(0) + s\,\mathcal{L}\{f'(t)\}(s)$$
$$= -f'(0) + s\left[-f(0) + sF(s)\right]$$
$$= s^2F(s) - sf(0) - f'(0)$$

Exercise Solve IVP with Laplace !! (i.e. without $x_p, x_H$, etc...)

$$* \begin{cases} x''(t) + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

Hard Theorem : Laplace transform is invertible (and so the inverse is also linear)

Answer : Write $\mathcal{L}\{x(t)\}(s) = X(s)$.

We know there is a unique sol'n to $*$.

get equality when Laplace both sides of the DE :

$$\mathcal{L}\{x''(t)\}(s) + 4\mathcal{L}\{x(t)\}(s) = 10\,\mathcal{L}\{\cos 3t\}(s)$$

$$s^2X(s) - s(2) - 1 + 4X(s) = 10\frac{s}{s^2+9}$$

$$X(s)(s^2+4) = \frac{10s}{s^2+9} + 1 + 2s$$

$$X(s) = \frac{10s}{(s^2+4)(s^2+9)} + \frac{1}{s^2+4} + \frac{2s}{s^2+4}$$

$$= 10s\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right]\left[\frac{1}{5}\right] + \frac{1}{s^2+4} + \frac{2s}{s^2+4}$$

$$X(s) = -\frac{2s}{s^2+9} + \frac{1}{s^2+4} + \frac{4s}{s^2+4}$$

Table!! $\boxed{x(t) = -2\cos 3t + \frac{1}{2}\sin 2t + 4\cos 2t}$ Magic!!

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $1/s$ |
| $e^{at}$ | $1/(s-a)$ |
| $f'(t)$ | $sF(s) - f(0)$ |
| $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ |
| $\cos kt$ | $\frac{s}{s^2+k^2}$ |
| $\sin kt$ | $\frac{k}{s^2+k^2}$ |
| $\vdots$ | |

we'll be filling this in! see e.g. book cover.

If we have time, continue building Laplace transform table:

Finish 1a: (Also 0!)

let $g(t)=\int_0^t f(\tau)\,d\tau$

so $\begin{cases} g'(t)=f(t) \\ g(0)=0 \end{cases}$

so $\underbrace{\mathcal{L}\{g'(t)\}(s)}_{=F(s)} = s\,G(s)-0$ so $G(s)=\frac{F(s)}{s}$

1b is numbered with 1a because derivatives on one side of the column correspond to multiplying the transform by variable, on the other!

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt$$

So what is $F'(s)$ ??

| $f(t)$ | $F(s)$ | for $t\geq 0$, $\lvert f(t)\rvert \leq C e^{Mt}$ |
|---|---|---|
| $f(t)$ | $\int_0^\infty e^{-st}f(t)\,dt$ | $s>M$ |
| $c_1f_1(t)+c_2f_2(t)$ | $c_1F_1(s)+c_2F_2(s)$ ($\mathcal{L}$ is linear) | |
| 0: $1$ | $1/s$ | $s>0$ |
| $e^{at}$ | $1/(s-a)$ | $s>\operatorname{Re} a$ |
| $\cos kt$ | $s/(s^2+k^2)$ | $\vdots$ |
| $\sin kt$ | $k/(s^2+k^2)$ | |
| $e^{at}\cos kt$ | $(s-a)/((s-a)^2+k^2)$ | |
| $e^{at}\sin kt$ | $k/((s-a)^2+k^2)$ | |
| 1a: $f'(t)$ | $sF(s)-f(0)$ | |
| $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ | |
| $f'''(t)$ | $s^3F(s)-s^2f(0)-sf'(0)-f(0)$ | |
| $\int_0^t f(\tau)\,d\tau$ | $F(s)/s$ | |
| 1b: $t\,f(t)$ | $-F'(s)$ | |
| $t^2f(t)$ | $F''(s)$ | |
| $t^3f(t)$ | $-F'''(s)$ | |
| $f(t)/t$ | $\int_s^\infty F(\sigma)\,d\sigma$ | |
| 2: $e^{at}f(t)$ | $F(s-a)$ | |
| $1$ | $1/s$ | |
| $t$ | $1/s^2$ | |
| $t^2$ | $2/s^3$ | |
| $t^n$ | $n!/s^{n+1}$ | $n\in\mathbb{N}$ |
| $t\,e^{at}$ | $1/(s-a)^2$ | |
| $t^ne^{at}$ | $n!/(s-a)^{n+1}$ | |
| $t\cos kt$ | $(s^2-k^2)/(s^2+k^2)^2$ | 3 resonance! |
| $\frac{t}{2k}\sin kt$ | $s/(s^2+k^2)^2$ | |
| $\frac{1}{2k^3}(\sin kt - kt\cos kt)$ | $1/(s^2+k^2)^2$ | |

MORE!!