

Math 2280-1
Monday April 12
6.4 Mechanical systems

①

rigid-rod pendulum:

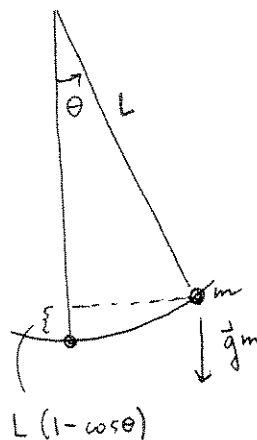
$$(1) \quad \theta''(t) + \frac{g}{L} \sin \theta = 0$$

We derived this DE using
KE + PE = constant

$$\frac{1}{2} m L^2 (\theta')^2 + mgL(1 - \cos \theta) = \text{const}$$

$$(2) \quad \frac{1}{2} L (\theta')^2 + g(1 - \cos \theta) = \tilde{\text{const}}$$

(and we took $\frac{d}{dt}(\%)$ to get (1))



For

$$x = \theta(t)$$

$$y = \theta'(t)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L} \sin x \end{bmatrix}$$

equil sol's $y = 0$
 $\sin x = 0: x = k\pi, k \in \mathbb{Z} (0, \pm 1, \pm 2, \dots)$

linearization:

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} \cos x & 0 \end{bmatrix}$$

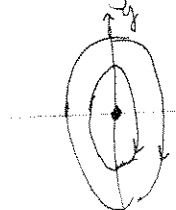
if $x = k\pi$ k even

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 + \left(\frac{g}{L}\right) = 0$$

$$\lambda = \pm i\sqrt{\frac{g}{L}}$$

linearization has stable center



(why clockwise?)

if $x = k\pi$, k odd

$$J = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 - \frac{g}{L} = 0$$

$$\lambda = \pm \sqrt{\frac{g}{L}} \quad \text{saddle}$$

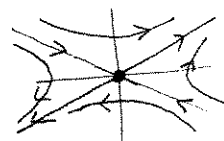
$$\lambda = \sqrt{\frac{g}{L}}$$

$$\lambda = -\sqrt{\frac{g}{L}}$$

$$\begin{bmatrix} -\sqrt{\frac{g}{L}} & 1 \\ \frac{g}{L} & -\sqrt{\frac{g}{L}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -1 \\ \sqrt{\frac{g}{L}} \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{\frac{g}{L}} \end{bmatrix}$$



fill in?

2

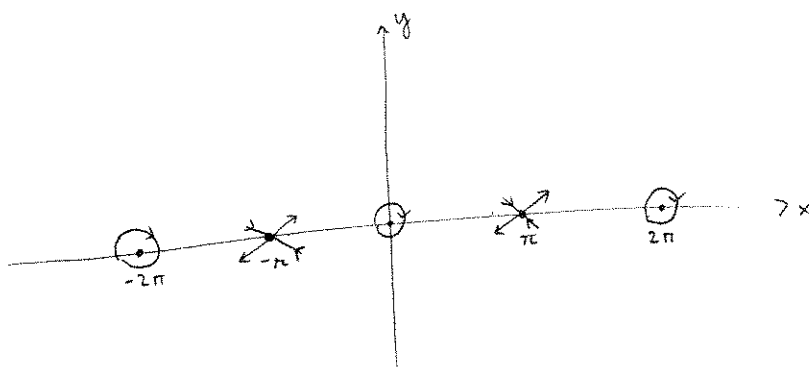
Hint: the trajectories must follow the level curves of the (scaled) total energy function (2), on page 1, i.e.

$$\frac{1}{2} L \dot{y}^2 + g(1 - \cos x) = \text{const} \quad \leftarrow \text{RHS is } 2\pi\text{-periodic in } x.$$

notice this function of (x, y)

attains its minimum value of zero, at all $(x, y) = (k\pi, 0)$
with k even
(so $\cos x = 1$)

and the Hessian matrix of this function is diagonal with positive diagonal entries, at these points, so the nearby trajectories for our system are (nearly) ellipses, for the non-linear system, so $(k\pi, 0)$ is stable center for the nonlinear problem, when k is even!



Add damping:

$$\ddot{\theta} + c \dot{\theta} + \frac{g}{L} \sin(\theta) = 0$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L} \sin x - cy \end{bmatrix}$$

same equilibria (iff $y=0$ & $\sin x=0$)

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} \cos x & -c \end{bmatrix}$$

$$|J - \lambda I| = -\lambda(-\lambda - c) + \frac{g}{L} \cos x$$

$$= \lambda^2 + c\lambda \pm \frac{g}{L} \quad \left(\begin{array}{l} +\frac{g}{L} \text{ if } x=k\pi \\ \text{even} \\ -\frac{g}{L} \text{ if } k \text{ odd} \end{array} \right)$$

$$\text{roots } \lambda = \frac{-c \pm \sqrt{c^2 \mp \frac{4g}{L}}}{2}$$

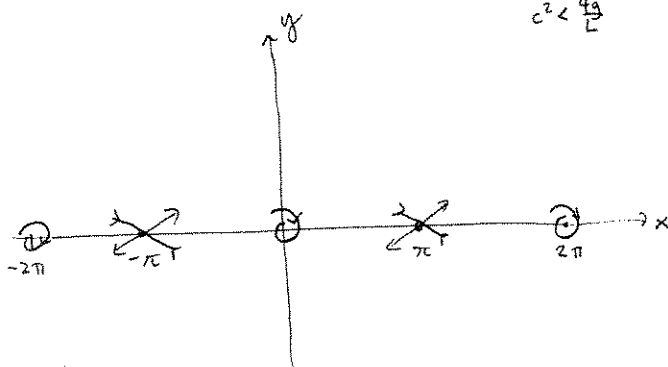
k odd $\Rightarrow$ saddle

k even, $c^2 < \frac{4g}{L} \Rightarrow$ stable spiral (underdamped)

$c^2 > \frac{4g}{L} \Rightarrow$ stable node (overdamped)

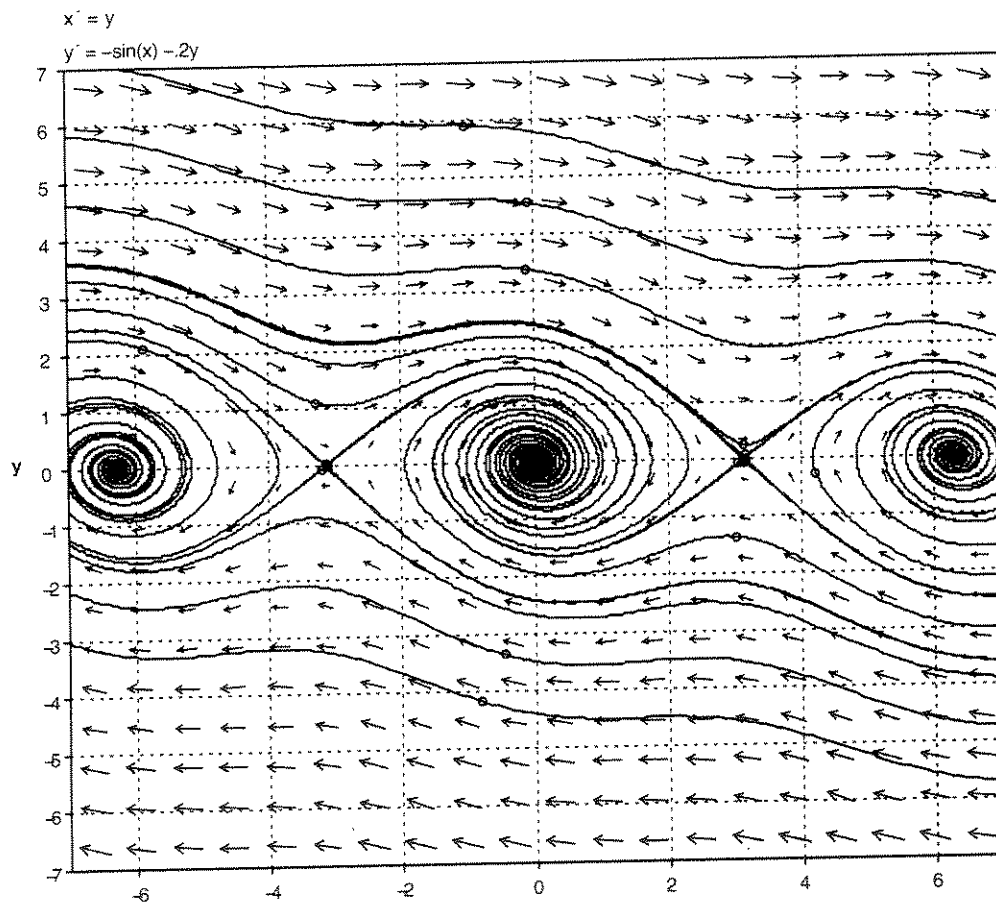
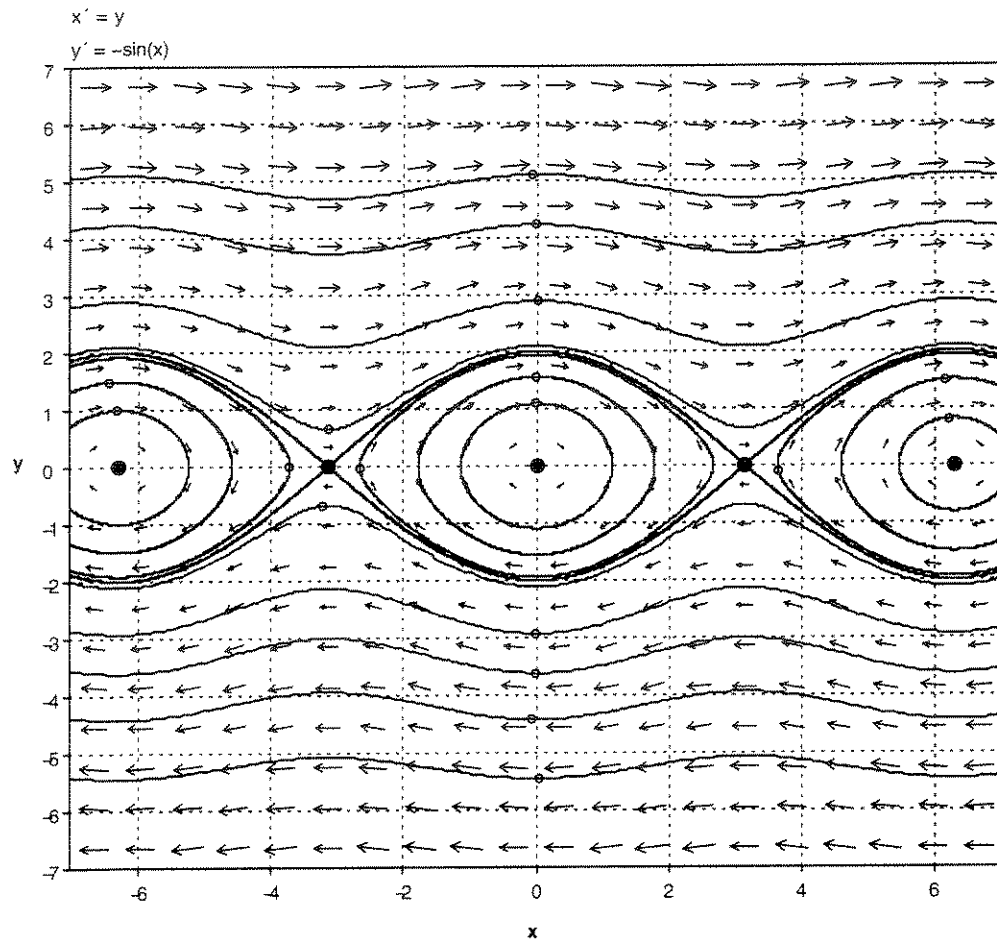
Fill in?

$$c^2 < \frac{4g}{L}$$



pictures for page 1-2:

3



nonlinear springs

$$m x'' = -c x' - k x + \beta x^3 \quad (\text{spring force is odd fun; } F(x) = -F(x))$$

 $\beta > 0$ "soft spring"

 $\beta < 0$ "hard spring"
Example 1 p 415

$$x'' + 4x - x^3 = 0 \quad (c=0, \beta=1)$$

has a $\mathcal{H}$ E fun which is constant along trajectories:

$$E = \frac{1}{2}(x')^2 + 2x^2 - \frac{x^4}{4} = \text{const}$$

$$\begin{aligned} x &= x \\ y &= x' \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ x^3 - 4x \end{bmatrix} \end{aligned} \quad \rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x^3 - 4x}{y} \quad \text{so } y dy = (x^3 - 4x) dx$$

$$\frac{1}{2} y^2 = \frac{x^4}{4} - 2x^2 + C$$

$$\text{equil: } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

you can do the linearization analysis & deduce the phase portrait page 553, and below
Would such a spring make sense physically, at least for large x ?

