Math 2280-1
PRACTICE EXAM SOLUTIONS
April 2006

1) Consider the following two-tank configuration. In tank one there is uniformly mixed volume of 200 gallons, and pounds of solute x(t). In tank two there is mixed volume of 100 gallons and pounds of solute y(t). Water is pumped into tank one at a constant rate of 10 gallons/minute from an outside source, and this water has a constant solute concentration of 3 pounds/gallon. Water is pumped from tank one to tank two at constant rate of 10 gallons/minute, and from tank two to the sewer, also at a rate of 10 gallons/minute. Initially the water in each tank is pure.
(This a cascade of two tanks, in fact this is problem #16 in section 5.6)

1a) Derive the system of first order differential equations which governs the process described above.
(10 points)

*the rate into and out of each tank is 10 gallons per minute. Thus 3*10=30 pounds of salt per minute are entering tank 1, and then 10*(x/200) pounds per minute are leaving. Those pounds are going into tank 2, and the outlet from tank 2 is taking out 10*(y/100) pounds per minute. This leads to the system:*

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 30 - \frac{1\,x}{20} \\ \frac{1\,x}{20} - \frac{1\,y}{10} \end{bmatrix}.$$

1b) The homogenous part of the system of differential equations above is

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{20} & 0 \\ \frac{1}{20} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find a fundamental matrix solution for this homogenous system.
(10 points)

Because the matrix is (lower) triangular, the eigenvalues are the two diagonal entries, -0.05, -0.1. For lambda = -0.1 we see by observation that [0,1] is an eigenspace basis. For lambda = -0.05 we seek solutions to the augmented system

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{20} & \frac{-1}{20} & 0 \end{bmatrix}$$

*from which we deduce an eigenvector of [1,1]. Thus we can make a FSM by putting our two linearly independent solutions (having form exp(lambda*t)*v) in as columns,:as follows:*

$$\begin{bmatrix} \mathbf{e}^{(-0.05\,t)} & 0 \\ \mathbf{e}^{(-0.05\,t)} & \mathbf{e}^{(-0.1\,t)} \end{bmatrix}$$

1c) Find the matrix exonential for the matrix in part 1b).

(10 points)

We can multiply the FSM from 1b by its inverse at t=0, on the right: It is quickest for 2 by 2 matrices to use the adjoint formula for the inverse:

$$\begin{bmatrix} e^{(-0.05\,t)} & 0 \\ e^{(-0.05\,t)} & e^{(-0.1\,t)} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e^{(-0.05\,t)} & 0 \\ e^{(-0.05\,t)} - e^{(-0.1\,t)} & e^{(-0.1\,t)} \end{bmatrix}$$

1d) Find a particular solution for the inhomogenous system in part 1a.

(10 points)

We try a particular solution of the form "k", where k is a constant vector. Plugging this into the differential equation leads to the matrix equation

$$0 = \begin{bmatrix} -\frac{1}{20} & 0 \\ \frac{1}{20} & -\frac{1}{10} \end{bmatrix}\begin{bmatrix} k1 \\ k2 \end{bmatrix} + \begin{bmatrix} 30 \\ 0 \end{bmatrix}$$

which has solution

$$\begin{bmatrix} k1 \\ k2 \end{bmatrix} = \begin{bmatrix} 600 \\ 300 \end{bmatrix}.$$

This makes sense because we expect the particular solution above to be the long-time solution, and as t-> infinity we would expect both tanks to have limit concentrations of 3 lbs/gallon.

1e) Solve the initial value problem in part 1a

(10 points)

The general solution is F(t)c + k, where F(t) is a FMS for the homogeneous equation, c is an unknown vector, and k is the vector we found in part 1d. We want initial values zero, so 0=F(0)c + k, or c=(F(0))^(-1)(-k). We can use either the FMS or the exponential matrix for F(t). If we use the FMS from 1b, we see that*

$$\begin{bmatrix} c1 \\ c2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -600 \\ -300 \end{bmatrix}$$

so that c1=-600 and c2=300. Thus IVP solution can be written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{(-0.05\,t)} & 0 \\ e^{(-0.05\,t)} & e^{(-0.1\,t)} \end{bmatrix}\begin{bmatrix} -600 \\ 300 \end{bmatrix} + \begin{bmatrix} 600 \\ 300 \end{bmatrix}$$

2) DERIVE either the variation of parameters formula using a general FMS, or the particular case of it

using matrix exponentials, for finding solutions to the inhomogeneous system of differential equations

$$\frac{dx}{dt} = Ax + \mathrm{f}(t).$$

(15 points)

Rather than repeat the derivation we did in class notes and in the book, I refer you to equations 15-29 on pages 361-363 of the text (section 5.6).

3) Consider the following configuration of springs, with positive displacements from equilibrium measured to the right, as indicated.

Mass 1 is to the left of mass 2, and there is a wall to the right of mass 2. There is a spring with constant k1 between these two masses. There is also a spring with constant k2 connecting mass 2 to the wall.

3a) Derive the system of second order differential equations which models this system. Assume that there are no external forces.

(5 points)

$$m_1 \left(\frac{d^2}{dt^2} \mathrm{x}(t) \right) = k_1 \, (y - x)$$

$$m_2 \left(\frac{d^2}{dt^2} \mathrm{y}(t) \right) = -k_1 \, (y - x) - k_2 \, y$$

3b) Assume that in appropriate units m1=2, m2=2, k1=4, k2=6. Show that in this case your system above reduces to

$$\begin{bmatrix} \dfrac{d^2 x}{dt^2} \\[2ex] \dfrac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2\,x + 2\,y \\ 2\,x - 5\,y \end{bmatrix}$$

(5 points)

This is easy to see since k1/m1=2, k1/m2=2, k2/m2=3.

3c) Find the general solution to the unforced system (7b).

(15 points)

For A defined as

$$A := \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$$

we find the eigenvalues and eigenvectors. The square roots of the opposites of the eigenvalues are the fundamental angular frequencies, the eigenvectors are the fundamental modes....of course you would do this part by hand, using the characteristic equation to find the eigenvalues, and then finding the eigenbases, but I have Maple going here!

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> eigenvects(A);
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$$[-1, 1, \{[2, 1]\}], [-6, 1, \{[1, -2]\}]$$

We deduce that the general solution is

$$\begin{bmatrix} \mathrm{x}(t) \\ \mathrm{y}(t) \end{bmatrix} = (c_1 \cos(t) + c_2 \sin(t)) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (c_3 \cos(\sqrt{6}\, t) + c_4 \sin(\sqrt{6}\, t)) \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3d) Assuming omega is not a natural frequence for the problem above, find a particular solution to the forced system

$$\begin{bmatrix} \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2\,x + 2\,y + \cos(\omega\, t) \\ 2\,x - 5\,y - \cos(\omega\, t) \end{bmatrix}.$$

(10 points)

*We try a particular solution of the form xp=cos(wt)**c**, where we find c by substituting xp(t) into the inhomogeneous DE. For*

$$c := \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

$$b := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We get the equation

$$-\omega^2 \cos(\omega\, t)\, c = \cos(\omega\, t)\, A\, c + \cos(\omega\, t)\, b$$

We divide by the scalar function cos(wt), and reduce to the matrix equation

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \omega^2 & 2 \\ 2 & -5 + \omega^2 \end{bmatrix} \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

Which we can solve with Cramer's rule or via an inverse matrix, yielding:

$$\begin{bmatrix} c1 \\ c2 \end{bmatrix} = \begin{bmatrix} \frac{3 - \omega^2}{(\omega^2 - 6)\,(\omega^2 - 1)} \\ \frac{\omega^2}{(\omega^2 - 6)\,(\omega^2 - 1)} \end{bmatrix}$$

4) Consider the following system of differential equations which is supposed to model two interacting species:

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 5x - x^2 - xy \\ \\ -2y + xy \end{bmatrix}.$$

4a) Would this system be modeling a coorperative, competetive, or predator-prey situation. Explain.

(5 points)

This is predator-prey, since the presence of predator y decreases the population growth rate of prey x and increases the growth rate for y. Also, in the absence of y, x grows logistically, whereas in the absence of x, the population y dies out

4b) Show that there are three equilibrium solutions to this system, namely [0,0], [5,0], [2,3].

(5 points)

We set the tangent vector field functions to zero and solve. They both factor, as follows

$$\begin{bmatrix} 5x - x^2 - xy \\ \\ -2y + xy \end{bmatrix} = \begin{bmatrix} x(5 - x - y) \\ \\ y(-2 + x) \end{bmatrix}$$

So there are potentially 4 critical points, where we require at least one of the factors in each expression to be zero. We can catalog these: If x=0, then y=0 will work (but -2+x=0 will not). If x is non-zero then if y=0 we need x=5, and if x=2 we need y=3. Thus we arrive at the collection of 3 points [0,0],[5,0],[2,3].

4c) Compute the linearized differential equation near each of the three equilibria from part (a). For each equilibrium solution use eigenvalue, eigenvector analysis to sketch a local phase portrait near the equilibrium solution, and indicate what type of equilibrium you are dealing with, and its stability characteristics.

(45 points)

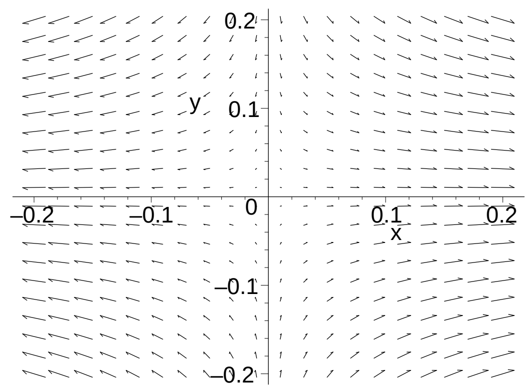
The derivative matrix for our tangent vector field is given by

$$J = \begin{bmatrix} 5 - 2x - y & -x \\ \\ y & -2 + x \end{bmatrix}$$

You get the matrix of the linearized differential equation at each equilbrium solution by plugging in the appropriate x and y values. For example, at [0,0] we get the matrix

$$\begin{bmatrix} 5 & 0 \\ \\ 0 & -2 \end{bmatrix}$$

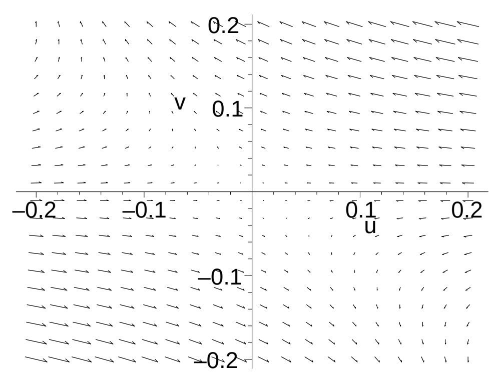
which has eigenvalue 5 (eigenvector e1=[1,0]) and eigenvalue -2 (eigenvector e2=[0,1]), so the origin is a saddle (always unstable), attracting along the y-axis and repelling along the x-axis). The phase portrait you draw would look something like



At [5,0] we get matrix

$$\begin{bmatrix} -5 & -5 \\ 0 & 3 \end{bmatrix}$$

Since the matrix is diagonal the eigenvalues are -5 and 3, so we have another unstable saddle. For the eigenvalue -5 and eigenvector is e1=[1,0]. For the eigenvalue 3 one obtains eigenvector [-5,8]. So this saddle attracts along the x-axis, and repulses in the direction of [-5,8]. Your phase portrait would look something like



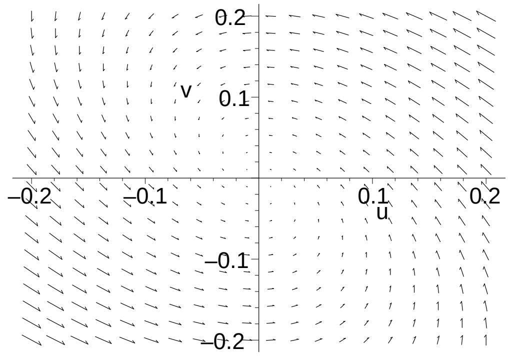
Finally, at [2,3] we get matrix

$$\begin{bmatrix} -2 & -2 \\ 3 & 0 \end{bmatrix}$$

which has characteristic polynomial

$$\lambda^2 + 2\,\lambda + 6$$

with complex roots -1 plus or minus the square root of 5, times i. So this point is a stable spiral. We can figure out how the spiral is rotating, by for example computing the tangent field at [1,0], which is [-2,3]. Thus the rotation is counterclockwise. Without further analysis you can't figure out the eccentricity of the spiral, but that wouldn't be necessary to answer this question. Your local phase portrait looks something like:



4d) Sketch the phase portrait (in the first quadrant) for the full non-linear system, using your information from part 4c. Explain what this means for the long-time behavior of solutions to this system.

(15 points)

This problem was taken from the homework problems of section 6.3, #11-13. After plotting the saddles at the origin and at [5,0] carefully (the attracting eigenvector at [0,0] is e2, the repulsing on is e1, the attracting eigenvector at [5,0] is e1, and the repulsing one is [-5,8], the stable spiral at [2,3] rotates counterclockwise, the most sensible way to piece together a global phase portrait for the first quadrant will give you a crude picture like Figure 6.3.14 on page 404. This means that if you start with ANY initial populations [x0,y0] in the interior of the first quadrant, the long time solutions will converge to the equilibrium at [2,3]!