Name...
I.D. number..

Math 2280-1
Practice Exam 1
February 13, 2006
SOLUTIONS

1) Consider the differential equation

$$\frac{dP}{dt} = -P^2 + 4\,P - 3$$

which models a certain population problem.

1a) Find the equilibrium solutions.

(4 points)

$$-P^2 + 4\,P - 3 = -[P-1]\,[P-3]$$

so the equilibrium solutions are P=1 and P=3.

1b) Sketch the phase diagram for this differential equation.

(4 points)

It's hard to draw this in Maple, so I'll describe it: the phase diagram is a copy of the P-axis, with the equilibrium solutions P=1 and P=3 highlighted. Since dP/dt is negative for P<1, there is an arrow pointing in the decreasing direction to the left of P=1. Since P is increasing for 1<P<3, there is an arrow pointing towards P=3 on this interval. Since P is decreasing for P>3, there is an arrow pointing towards 3 on this interval.

1c) Which of the equilibrium solutions are stable? Which are unstable?

(4 points)

From the phase diagram we deduce that P=1 is unstable and P=3 is stable.

1d) Sketch the slope field for this differential equation. Onto the slope field sketch graphs of the solutions to the three initial value problems with P(0)=0, P(0)=2, P(0)=4. (You don't need formulas for the solutions to make the sketches!)

(6 points)

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> restart:
  with(DEtools):deqtn:=diff(P(t),t)=-P(t)^2+4*P(t)-3;
  DEplot(deqtn,P(t),t=0..3,P=-2..4,{[P(0)=0],[P(0)=2],
    [P(0)=4]},arrows=line,color=black, linecolor=black,
    stepsize=.2, dirgrid=[30,30]);
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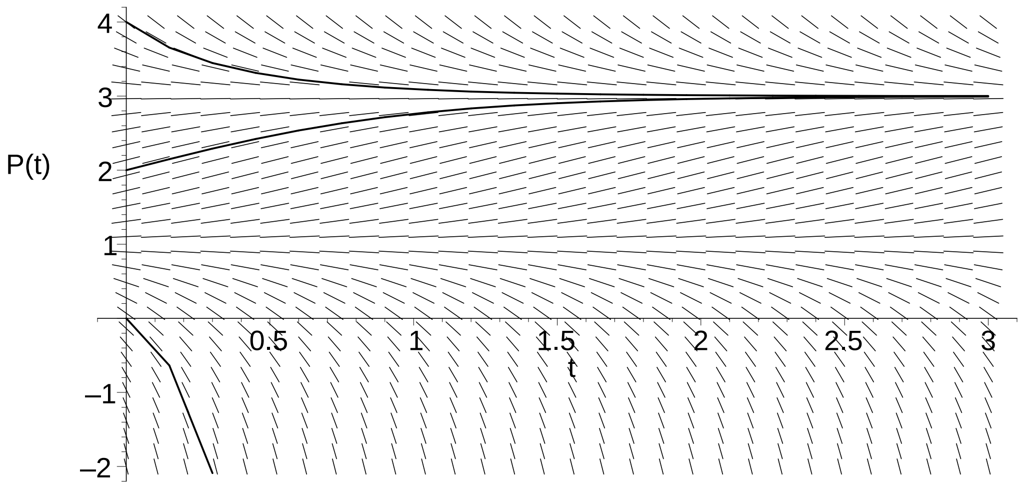
$$deqtn := \frac{d}{dt}\,\mathrm{P}(t) = -\mathrm{P}(t)^2 + 4\,\mathrm{P}(t) - 3$$



1e) Give a population model which leads to differential equations of this type. Be as precise as you can, so that you can account for the signs of all three terms on the right-hand side of the differential equation. (4 points)

This could be a logistic equation with harvesting, since

$$\frac{dP}{dt} = -P^2 + 4\,P$$
$$= P\,[4 - P]$$

is a logistic DE, with carrying population 4 (units). In the case of our DE, we would be harvesting 3 units of population per time unit, leading to

$$\frac{dP}{dt} = -P\,[4 - P] - 3.$$

1f) Find a explicit solution to the initial value problem for this differential equation, with P(0)=2. Verify that your limiting population agrees with what your sketch predicted in part 1b). (13 points)

$$\frac{DP}{(P-3)\,(P-1)} = -dt$$

Do partial fractions:

$$\frac{1}{(P-3)\,(P-1)} = \frac{1}{2}\,\frac{1}{P-3} - \frac{1}{2}\,\frac{1}{P-1}$$

Integrate:

$$\frac{1}{2}\ln(|P-3|) - \frac{1}{2}\ln(|P-1|) = -t + C$$

Exponentiate:

$$\left|\frac{P-3}{P-1}\right| = \mathbf{e}^{(-2\,t+c)}$$

plug in initial value

$$\ln(|-1|) = e^{c}$$

So c=0. Then remove absolute value sign carefully, because P(0)=2:

$$\frac{P-3}{P-1} = -e^{(-2\,t)}$$

$$P-3 = -(P-1)\,e^{(-2\,t)}$$

$$P\,(1+e^{(-2\,t)}) = e^{(-2\,t)} + 3$$

$$P = \frac{e^{(-2\,t)}+3}{1+e^{(-2\,t)}}\,.$$

Notice that P(t) converges to P=3 as t goes to infinity, as predicted by the phase diagram and slope field.

2) Consider a brine tank which holds 15,000 gallons of continuously-mixed liquid. Let x(t) be the amount of salt (in pounds) in the tank at time t. The in-flow and out-flow rates are both 150 gallons/hour, and if the concentration of salt flowing in is 1 pound per 10 gallons of water.

2a) Explain how the information above leads to the differential equation

$$\frac{dx}{dt} + 0.01\,x = 15$$

(5 points)

The rate of change of x(t) is the sum of the rate at which x comes into the tank, minus the rate at which it leaves. The rate in is the concentration in times the volume rate in, i.e. (0.1)(150) = 15 pounds/hour. The volume is constant 15,000 gallons since water leaves at the same rate it enters. Hence the concentration out is the quotient x(t)/15,000 pounds per gallon, and so salt is leaving at a rate of (x(t)/15,000)*150 pounds per hour. This is the differential equation*

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> deqtn:=diff(x(t),t) = 15 - (x(t)/15000)*150;
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$$deqtn := \frac{d}{dt}\,\mathrm{x}(t) = 15 - \frac{1}{100}\,\mathrm{x}(t)$$

which simplifies to

$$\frac{dx}{dt} + 0.01\,x = 15$$

2b) Solve the initial value problem for this differential equation, assuming that at time t=0 there is no salt in the water. You may use either chapter 1 or chapter 3 techniques.

(10 points)

$$e^{(0.01\,t)}\left(\left(\frac{d}{dt}\,\mathrm{x}(t)\right) + 0.01\,\mathrm{x}(t)\right) = 15\,e^{(0.01\,t)}$$

Integrate:

$$e^{(0.01\,t)}\,\mathrm{x}(t) = 1500.000000\,e^{(0.01\,t)} + C$$

solve for x:

$$\mathrm{x}(t) = 1500 + C\,e^{(-0.01\,t)}$$

Solve for C with initial value x(0)=0:

$$0 = 1500 + C$$
$$C = -1500$$
$$x(t) = 1500 - 1500\, \mathbf{e}^{(-0.01\, t)}.$$

2c) What is the limiting amount of salt as t approaches infinity?

(5 points)

As t-> infinity the exponential term dies out and we get 1500 pounds as the limiting amount. We expect this because we expect the limiting concentration to be the inflow concentration of 0.1 pounds/gallon, and there are 15,000 gallons in the tank.

3a) Consider the differential equation

$$\frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 5\, x = 0$$

Find the general solution to this differential equation. If this was modeling a spring problem, what kind of damping would we have?

(15 points)

The characteristic polynomial is

$$p(r) = r^2 + 4\, r + 5.$$

which has roots

$$-2 + i,\ -2 - i$$

Thus the general solution is

$$x(t) = \mathbf{e}^{(-2\, t)} (A \cos(t) + B \sin(t)).$$

Since the roots are complex, this is an underdamped system.

3b) Solve the initial value problem for the differential equation above, for x(0)=2, Dx(0)=-2.

(10 points)

$$\frac{dx}{dt} = \mathbf{e}^{(-2\, t)} (-2\, A \cos(t) - 2\, B \sin(t) - A \sin(t) + B \cos(t))$$

Plugging in initial data yields

$$2 = A$$
$$-2 = -2\, A + B$$

which has solution

$$A = 2$$
$$B = 2$$

So our solution is

$$x(t) = \mathbf{e}^{(-2\, t)} (2 \cos(t) + 2 \sin(t)).$$

3c) Write your solution to 3b) in the form of a time-varying amplitude times cos(wt-a).

(10 points)

*We use the right triangle with horizontal displacement A, vertical displacement B. Then the hypotenuse is C, and the angle between C and A is alpha. So we have C=2*sqrt(2) and alpha is the arctan of 1, ie Pi/4.*

$$x(t) = 2\sqrt{2}\,\mathbf{e}^{(-2\,t)}\cos\left(t - \frac{\pi}{4}\right)$$

4) We know that any expression

$$A\cos(\omega\,t) + B\sin(\omega\,t)$$

Can be rewritten as

$$C\cos(\omega\,t - \alpha),$$

and you probably memorized the formulas for this exam.

4a) Use the addition angle formula for cosine to rederive the formulas for C and alpha in terms of A and B.

(5 points)

$$C\cos(\omega\,t - \alpha) = C\,(\cos(\omega\,t)\cos(\alpha) + \sin(\omega\,t)\sin(\alpha))$$

So, equating coefficients we deduce that

$$C\cos(\alpha) = A$$
$$C\sin(\alpha) = B$$

This leads to the ABC triangle discussed in class and above, i.e.

$$C = \sqrt{A^2 + B^2}$$
$$\cos(\alpha) = \frac{A}{C}$$
$$\sin(\alpha) = \frac{B}{C}$$

4b) Find C and alpha so that

$$\sin(t) + 2\cos(t) - \sqrt{2}\cos\left(t + \frac{1}{4}\pi\right) = C\cos(t - \alpha)$$

(5 points)

$$\sin(t) + 2\cos(t) - \sqrt{2}\left(\cos(t)\,\mathrm{Cos}\left(\frac{1}{4}\pi\right) - \sin(t)\,\mathrm{Sin}\left(\frac{1}{4}\pi\right)\right) = C\,\mathrm{Cos}(t - \alpha)$$
$$\sin(t) + 2\cos(t) - \sqrt{2}\left(\frac{1}{2}\cos(t)\sqrt{2} - \frac{1}{2}\sin(t)\sqrt{2}\right) = C\,\mathrm{Cos}(t - \alpha)$$

$$\cos(t) + 2\sin(t) = C\,\mathrm{Cos}(t - \alpha)$$

So we have A=1, B=2. Hence

$$C = \sqrt{5}$$
$$\alpha = \arctan(2)$$

5) (Well, I ran out of points, but here's another problem): Solve

$$\frac{d^3\,x}{dt^3} + \frac{2\,d^2\,x}{dt^2} + \frac{dx}{dt} = 0$$

With initial condition x(0)=1, Dx(0)=1, D^2(x)(0)=0.

(15 points)

$$\mathrm{p}(r) = r^3 + 2\,r^2 + r$$
$$\mathrm{p}(r) = r\,(r^2 + 2\,r + 1)$$
$$\mathrm{p}(r) = 2\,r\,(r + 1)^2$$

So the general solution is

$$\mathrm{x}(t) = c_1 + c_2\,\mathbf{e}^{(-t)} + c_3\,t\,\mathbf{e}^{(-t)}.$$

When you compute the first and second derivatives of x(t) and plug in the initial data you get a system

$$c_1 + c_2 = 1$$
$$-c_2 + c_3 = 1$$
$$c_2 - 2\,c_3 = 0$$

which you solve for $c_1 = 3$, $c_2 = -2$, $c_3 = -1$, *so that*

$$\mathrm{x}(t) = 3 - 2\,\mathbf{e}^{(-t)} - t\,\mathbf{e}^{(-t)}.$$

END!!