Name..
I.D. number...

Math 2280-1
Practice Final Exam

April 2006

The exam is closed-book and closed-note. You may use a scientific calculator, but not one which does linear algebra, differential equations, or symbolic integration computations. You will be provided with a Laplace transform table and any non-standard integrals you need. This exam counts for 30% of your final grade, and is written so that there are 150 points possible. Point values are indicated to the right of each problem. Good Luck!!
(NOTE: AS USUAL THIS IS ONLY A SAMPLE OF THE KINDS OF QUESTIONS THAT MAY BE ASKED. IT IS NOT INTENDED TO BE INCLUSIVE OF ALL POSSIBILITIES!)

1) A motorboat weighs 32000 pounds and its motor provides a thrust force of 5000 pounds. Assume the water provides a linear drag, with resistance force equal to 100 pounds for each foot per second of the speed v of the boat.
1a) For the description above, derive the first order differential equation

$$1000\frac{dv}{dt} = 5000 - 100\,v$$

for the velocity of the boat at time t.
A motorboat weighing 32000 pounds has a mass of 1000 slugs, since weight = force = mg, and g=32 ft/sec^2. So the equation above is just Newton's law that mass times acceleration equals net forces, in this case the sum of the motorboat engine propulsion (5000 pounds), and the linear drag of -100v pounds.

(3 points)

1b) If the boat started from rest predict its limiting speed, based on the an analysis of the equilibrium solutions to the differential equation above?

(2 points)

The equilibrium solution for velocity will be a constant (velocity) solution, obtained by setting the right side equal to zero, i.e. 5000-100v=0, or v=50 ft/sec. If we drew the phase diagram for velocity we would see that this equilibrium solution is stable, and that the solution to the IVP with v(0)=0 would approach this v as t->infinity.

1c) Find the explicit solution to the initial value problem for this differential equation, with v(0)=0. Their are FOUR ways you can do this problem (first order linear, separable, general linear, Laplace transform). Solve it each way! (5 points per method).

(20 points)

(i) First order linear:

$$\frac{dv}{dt} + 0.1\,v = 5$$

$$\mathbf{e}^{(0.1\,t)}\left(\frac{dv}{dt} + 0.1\,v\right) = 5\,\mathbf{e}^{(0.1\,t)}$$

$$\mathbf{e}^{(0.1\,t)}\,v = 50\,\mathbf{e}^{(0.1\,t)} + C$$

Since v(0)=0 we deduce C=-50:

$$\mathbf{e}^{(0.1\,t)}\,v = 50\,\mathbf{e}^{(0.1\,t)} - 50$$

$$v = 50 - 50\,\mathbf{e}^{(-0.1\,t)}$$

Note, the limiting speed is indeed equal to 50 ft/sec.

(ii) Separable:

$$\frac{dv}{dt} = 5 - 0.1\,v$$

$$\frac{dv}{5 - 0.1\,v} = dt$$

$$-10\,\ln(|-5 + 0.1\,v|) = t + C$$

$$\ln(|-5 + 0.1\,v|) = -0.1\,t + C_1$$

$$5 - 0.1\,v = C_2\,\mathbf{e}^{(-0.1\,t)}$$

Since v(0)=0, we deduce

$$5 - 0.1\,v = 5\,\mathbf{e}^{(-0.1\,t)}$$

$$v = 50 - 50\,\mathbf{e}^{(-0.1\,t)}.$$

(iii) general linear:

$$\frac{dv}{dt} + 0.1\,v = 5$$

The solution to the homogenous problem

$$\frac{dv}{dt} + 0.1\,v = 0$$

is

$$v_H(t) = C\,\mathbf{e}^{(-0.1\,t)}$$

(follows immediately from the characterstic polynomial.) A particular solution to the non-homogeneous equation is the constant

$$v_p(t) = 50$$

so the full solution to the DE is

$$\mathrm{v}(t) = 50 + C\,\mathbf{e}^{(-0.1\,t)}$$

and since v(0)=0, it must be that C=-50, and we get the same solution as in parts (i),(ii).

(iv) Laplace transform:

$$\frac{dv}{dt} + .1\,v = 5$$

$$s\,\mathrm{V}(s) + 0.1\,\mathrm{V}(s) = \frac{5}{s}$$

$$\mathrm{V}(s) = \frac{5}{s\,(s + 0.1)}$$

$$\mathrm{V}(s) = \frac{50}{s} - \frac{50}{s + 0.1}$$

$$v(t) = 50 - 50\,\mathbf{e}^{(-0.1\,t)} \quad !!!!!$$

2) Consider the undamped forced oscillator problem

$$\frac{d^2 x}{dt^2} + 9\,x = 2\cos(\omega\,t)$$

2a) For what value of (positive) omega do you expect resonance?

(3 points)

The natural angular frequency for this system is the square root of 9, i.e. 3, and that is the value of ω *which will cause resonance.*

2b) Find the general solution to this differential equation, in the case when there is no resonance.

(12 points)

I could do this Laplace transform, but I'll use Chapter 3 instead:

We try for a particular solution of the form $x_p(t) = A\cos(\omega\,t)$. *Plugging this candidate solution into the differential equation (and then dividing by* $\cos(\omega\,t)$, *gives us the algebraic equation for the unknown coefficient A:*

$$-A\,\omega^2 + 9\,A = 2$$

$$A = 2\,\frac{1}{9 - \omega^2}$$

So the general solution to this problem is the sum of this particular solution with the general solution to the homogeneous equation, i.e.

$$x(t) = c_1\cos(3\,t) + c_2\sin(3\,t) + \frac{2\cos(\omega\,t)}{9 - \omega^2}$$

3) Consider the following two-tank configuration: In tank one there is uniformly mixed volume of V1 gallons, and pounds of solute x(t). In tank two there is mixed volume of V2 gallons and pounds of solute y(t). Water us pumped into tank one at a constant rate of ri gallons/minute from an outside source, and this water has a constant solute concentration of ci pounds/gallon. Water is pumped from tank one to tank two at constant rate of r1 gallons/minute, from tank two to tank one at constant rate r2 gallons/minute, and out of the tank system (from tank 2) at constant rate ro gallons/minute.

3a) Write the system of first order differential equations which governs the process described above. Do not try to solve these DE's.

(3 points)

$$\frac{dx}{dt} = r_i\,c_i - \frac{r_1\,x}{V_1} + \frac{r_2\,y}{V_2}$$

$$\frac{dy}{dt} = -\frac{r_o\,y}{V_2} - \frac{r_2\,y}{V_2} + r_1\left(\frac{x}{V_1}\right)$$

$$\frac{dV_1}{dt} = r_i - r_1 + r_2$$

$$\frac{dV_2}{dt} = r_1 - r_2 - r_o$$

3b) Suppose that ri=ro=0, so that no water is flowing into or out of the system. Suppose further that r1=r2, so that V1 and V2 are constant. Set V1=100 gallons, V2=50 gallons, and let r1=r2=100 gallons per hour. Show that in this case the general system you derived in 3a) reduces to the first order system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(2 points)

r_i=r_o= 0, and r_1=r_2, so we see that $\frac{dV_1}{dt}$ *and* $\frac{dV_2}{dt}$ *are zero, so* V_1 *and* V_2 *stay constant. Since* r_1=r_2 *= 100 gallons/hour, and since* V_1*=100 gallons, and* V_2*= 50, our equations from part (a) become:*

$$\frac{dx}{dt} = -x + 2y$$

$$\frac{dy}{dt} = -2y + x$$

which becomes the DEqtn above, when written in matrix form.

3c) Use Laplace transform to solve the initial value problem for this tank system:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(10 points)

(You could also have been asked to solve this using chapter 5 techniques.) We take the Laplace transform of this IVP, which yields

$$\begin{bmatrix} s\,X(s) \\ s\,Y(s) - 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1-s & 2 \\ 1 & -2-s \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \left[\frac{1}{s\,(s+3)} \right] \begin{bmatrix} -2-s & -2 \\ -1 & -1-s \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \dfrac{6}{s(s+3)} \\ \dfrac{3+3s}{s(s+3)} \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{s} - \dfrac{2}{s+3} \\ \dfrac{1}{s} + \dfrac{2}{s+3} \end{bmatrix}$$

Taking inverse Laplace transform, we deduce

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 - 2\,\mathbf{e}^{(-3t)} \\ 1 + 2\,\mathbf{e}^{(-3t)} \end{bmatrix}.$$

4) Consider a configuration of two walls, 3 masses and four springs in a linear array, see Figure 5.3.1 page 315. The spring constants, from the left, are k_1, k_2, k_3, k_4. The mass constants, also from the left are m_1, m_2, m_3. As usual, we measure the displacements of the masses from equilibrium by x_1, x_2, x_3, and we choose the positive direction to be to the right.

4a) Find choices of the spring constants and the masses so that the second order system satisfied by the displacements is given by

$$\begin{bmatrix} \dfrac{d^2 x_1}{dt^2} \\ \dfrac{d^2 x_2}{dt^2} \\ \dfrac{d^2 x_3}{dt^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(3 points)

The equations which govern this system are

$$m_1 \left[\frac{dx_1}{dt}\right] = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 \left[\frac{dx_2}{dt}\right] = -k_2 (x_2 - x_1) + k_3 (x_3 - x_2)$$

$$m_3 \left[\frac{dx_3}{dt}\right] = -k_3 (x_3 - x_2) - k_4 x_3$$

and we see that if we take all the masses =1 and all the spring constants equal to 1, we get the second *order matrix system for part a.*

4b) Find the general solution to the system above. Hint: Use the information that the eigenvectors of the matrix A above are given by

```
> eigenvects(A);
```

$$[-2, 1, \{[-1, 0, 1]\}], [-2+\sqrt{2}, 1, \{[1, \sqrt{2}, 1]\}], [-2-\sqrt{2}, 1, \{[1, -\sqrt{2}, 1]\}]$$

(6 points)

Since this is a second order system of DE's we can read off the six-dimensional general solution from the eigenvectors. The angular frequencies are the square roots of the negatives of the matrix eigenvalues:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (c_1 \cos(\sqrt{2}\, t) + c_2 \sin(\sqrt{2}\, t)) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (c_3 \cos(\sqrt{2-\sqrt{2}}\, t) + c_4 \sin(\sqrt{2-\sqrt{2}}\, t)) \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$+ (c_5 \cos(\sqrt{2+\sqrt{2}}\, t) + c_6 \sin(\sqrt{2+\sqrt{2}}\, t)) \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

4c) Describe the three fundamental modes of this system. Which vibrates the quickest, and which vibrates the slowest?

(6 points)

The first fundamental mode indicated above, with angular frequency equal to sqrt(2), about 1.41, has masses 1 and 3 oscillating in opposition (out of phase), with equal amplitudes. The second fundamental mode above has all three masses oscillating in parallel, with the middle mass having amplitude sqrt(2) times bigger than the other two mass amplitudes. This mode has the slowest angular frequency, namely omega=sqrt(2-sqrt(2)), which is about 0.76 radians per second. The third mode has the middle mass oscillating out of phase, relative to the two outside masses. It's amplitude is sqrt(2) times theirs. Its angular frequency is the fastest, namely omega = sqrt(2+sqrt(2)), which is about sqrt(3.4), i.e. about 1.85 radians per second.

5) Consider the system of differential equations below which models two populations x(t), y(t):

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -x^2 - x\,y + 5\,x \\ -y^2 + x\,y + 3\,y \end{bmatrix}$$

5a) Maybe we should call this system the ''omnivorous predator-prey'' model, since one of the populations seems to deplete the other one but not be totally reliant on it for survival. Perhaps x(t) and y(t) are measuring how many thousands of each animal are present at time t years. Find all four equilibrium solutions which exist for this system of differential equations.

(8 points)

We get equilibrium solutions when dx/dt and dy/dt are zero, i.e. when the right side is zero. This leads to the equations

$$x(-x-y+5)=0$$
$$y(-y+x+3)=0$$

which has solutions x=0, y=0; x=0, y=3; y=0, x=5;, and finally, x=1,y=4. That is, we get the points

$$\left\{\begin{bmatrix}0\\3\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

5b) Classify the stability and type of each of the four equilibrium solutions from part (5a).

(16 points)

The derivative (Jacobian) matrix which we use is

$$\begin{bmatrix}\frac{\partial}{\partial x}\mathrm{F}(x,y) & \frac{\partial}{\partial y}\mathrm{F}(x,y)\\ \frac{\partial}{\partial x}\mathrm{G}(x,y) & \frac{\partial}{\partial y}\mathrm{G}(x,y)\end{bmatrix}=\begin{bmatrix}-2x-y+5 & -x\\ y & -2y+x+3\end{bmatrix}$$

At [0,0] we get

$$J:=\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}$$

which is an unstable node (source), with eigenvectors e1 (eval = 5) and e2 (eval =3).

At [0,3] we get

$$J:=\begin{bmatrix}2 & 0\\ 3 & -3\end{bmatrix}$$

```
> eigenvects(J);
```

$$\left[2,1,\left\{\left[\frac{5}{3},1\right]\right\}\right],[-3,1,\{[0,1]\}]$$

which has eigenvalues -3 and 2 (so is a saddle, i.e. unstable)....the -3 eigenbasis is [0,1], and the +2 eigenbasis is [5,-3].

At [5,0] the Jacobian matrix is given by

$$J:=\begin{bmatrix}-5 & -5\\ 0 & 8\end{bmatrix}$$

```
> eigenvects(J);
```

$$[-5,1,\{[1,0]\}],\left[8,1,\left\{\left[1,\frac{-13}{5}\right]\right\}\right]$$

which has eigenvalues -5 (evect = e1), and 8 (evect basis [-5,13]). So this point is an (unstable) saddle.
Finally, at [1,4] we have

$$J:=\begin{bmatrix}-1 & -1\\ 4 & -4\end{bmatrix}$$

```
> eigenvects(J);
```

$$\left[-\frac{5}{2}+\frac{1}{2}I\sqrt{7}, 1, \left\{\left[1, \frac{3}{2}-\frac{1}{2}I\sqrt{7}\right]\right\}\right], \left[-\frac{5}{2}-\frac{1}{2}I\sqrt{7}, 1, \left\{\left[1, \frac{3}{2}+\frac{1}{2}I\sqrt{7}\right]\right\}\right]$$

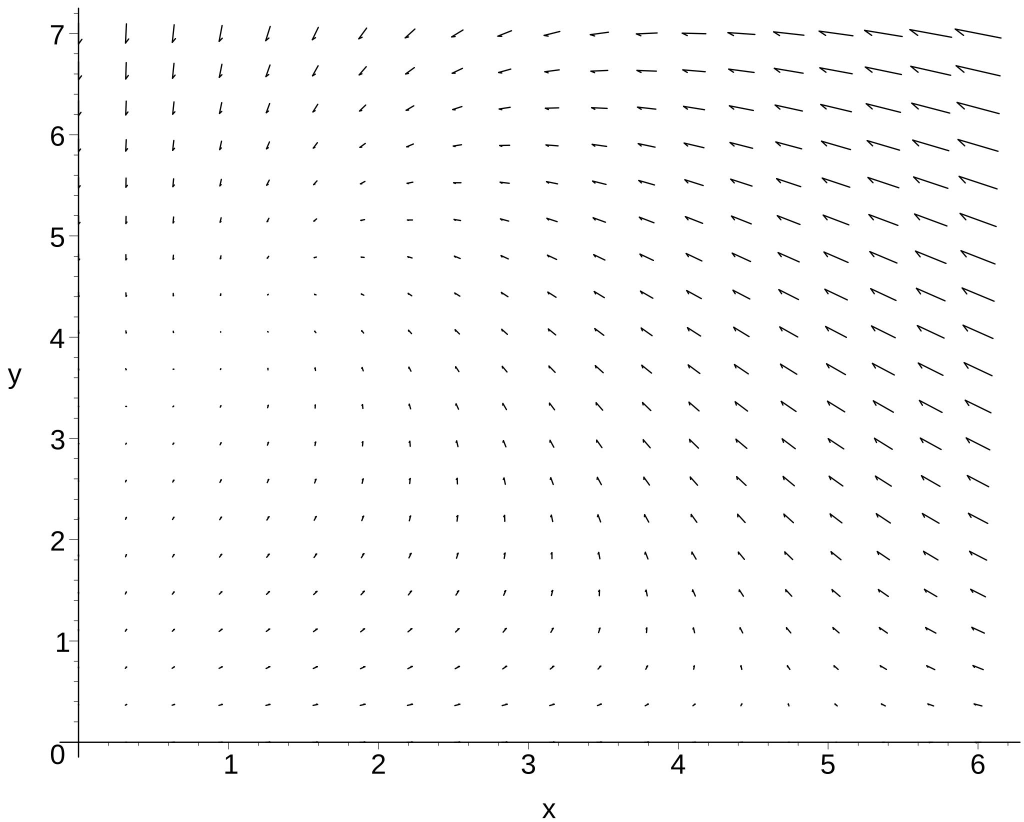
We only need to know that the eigenvalues are complex, and have negative real part. Therefore the interior equilibrium point is a stable spiral. It actually spirals counterclockwise, as you can determine by looking at the columns of the matrix J

5c) Make a rough sketch of the phase field in the first quadrant which is consistent with your information from part (5b). (So, for example, if you are sketching saddle points you needn't worry about exactly what the eigendirections are.) Assuming that both populations start initially with positive values, deduce the possible limiting populations as time aproaches infinity.

(6 points)

If you plot the local (linearized) pictures near each equilibrium point, taking into account the saddle pictures at [5,0] and [0,3], and which are the negative and positive eigenvectors, as well as the node at the origin and the stable interior spiral, you will get a picture which is consistent with the fieldplot below. I would expect you todraw in some sample trajectories, rather than the tangent vector field. It sure looks like all initial populations in which both species are represented converge to the stable equilibrium of coexistence.

```
> with(plots):
  fieldplot([-x^2-x*y+5*x,-y^2+x*y+3*y],x=0..6,y=0..7,color=black);
```



6a) Let f be a 2L-periodic function. Write down the Fourier series for f, and write down the formulas for the Fourier coefficients as well.

(8 points)

The Fourier series for f is

$$f := \frac{1}{2} a_0 + \left(\sum_{n=1}^{\infty} a_n \cos\left(\frac{n \pi t}{L}\right) \right) + \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n \pi t}{L}\right) \right)$$

where the Fourier coefficients are computed by

$$a_n := \left[\frac{1}{L}\right] \int_{-L}^{L} \mathrm{f}(t) \cos\left(\frac{n \pi t}{L}\right) dt$$

$$b_n := \left[\frac{1}{L}\right] \int_{-L}^{L} \mathrm{f}(t) \sin\left(\frac{n \pi t}{L}\right) dt$$

6b) Let f(x) be the period 2 function obtained by taking the odd extension of the function which equals x(1-x) on the interval [0,1]. Derive the Fourier series for f:

(7 points)

$$f(x) = \sum_{n=odd} \left(8 \frac{\sin(n \pi x)}{n^3 \pi^3} \right)$$

You may wish to use the integration formulas

$$\int x \sin(n \pi x)\, dx = \frac{\sin(n \pi x) - n \pi x \cos(n \pi x)}{n^2 \pi^2}$$

$$\int x^2 \sin(n \pi x)\, dx = \frac{-n^2 \pi^2 x^2 \cos(n \pi x) + 2 \cos(n \pi x) + 2 n \pi x \sin(n \pi x)}{n^3 \pi^3}$$

Since f is the odd extension it is an odd function, so all cosine coefficients will be zero. hence we need only compute sine coefficients, i.e. a sine series. To get bn we need to evaluate the first antiderivative above between zero and one, and then subtract off the second one, evaluated between the same endpoints. This is because f(x)=x-x^2. This answer should be multiplied by 2/L=2, since we are doing a sine series. Now, in the antidifferentiation formulas, all sine terms disappear at both x=0 and x=1, since sine is zero for integer multiples of pi. So we are left with

$$-\frac{\cos(n \pi)}{n \pi} - \frac{-n^2 \pi^2 \cos(n \pi) - 2 + 2 \cos(n \pi)}{n^3 \pi^3}$$

note that the first term is exactly cancelled by the second, so we are left with

$$-\frac{2 \cos(n \pi) - 2}{n^3 \pi^3}$$

which we recognize as giving us

$$4 \frac{1}{n^3 \pi^3}$$

in the case n is odd, and zero otherwise. Multiplying by 2/L=2 gives the desired result.

7a) Derive all possible product solutions u(x,t)=X(x)T(t) to the heat equation

$$u_t = k\, u_{xx}$$

with the ‘‘fixed boundary temperature’’ assumption that u(0)=u(L)=0.

(7 points)

Set u(x,t)=X(x)T(t), and substitute it into the heat equation. Please see the discussions in our class notes (apr19.pdf) and in the heat equation section of our text. The answer we got was a sequence of functions satisfying the two boundary conditions, namely for each counting number n,

$$\sin\left(\frac{n \pi x}{L}\right) \mathbf{e}^{\left(-\frac{k n^2 \pi^2 t}{L^2}\right)}$$

7b) Solve the initial boundary-value problem for the heat equation, where the initial temperature of a rod on the interval 0<x<1 is given by f(x)=x(1-x), and where the endpoint temperatures are held at temperature zero for positive time values. Write the solution for general heat diffusivity k. (Hint: the sine series for f was given in a previous problem.)

(8 points)

We use (infinite) superposition for this homogeneous PDE, the separated solutions from part (a), and the Fourier series from problem 6!

$$u(x, t) = 8\,\frac{\displaystyle\sum_{n=odd} \frac{\sin(n\,\pi\,x)\,\mathbf{e}^{(-k\,t\,n^2\,\pi^2)}}{n^3}}{\pi^3}$$

8) Consider the wave equation

$$u_{tt}(x, t) = 9\,u_{xx}(x, t)$$

on the interval 0<x<1, for positive time, and consider the initial boundary value problem where the endpoints are fixed (u(0,t)=u(1,t)=0), where the initial profile is given by the function $u(x, 0) = f(x) = \sin(\pi\,x)$, and where the initial velocity is $u_t(x, 0) = 0$. Find the solution $u(x, t)$.

(10 points)

This problem doesn't use Fourier series, it just uses the separation of variables solutions to the wave equation, (for speed a=3), which
satisfy the fixed endpoint boundary conditions, and have non-zero initial velocity (t-derivative), namely

$$\sin(n\,\pi\,x)\cos(3\,n\,\pi\,t)$$

For us, n=1, so our solution is simply

$$u(x, t) = \sin(\pi\,x)\cos(3\,\pi\,t)\,!$$

9) The existence and uniqueness theorem for linear first order systems of differential equations says that if A(t) is a continuous, n by n matrix-valued function of t, and f(t) is a continuous n-vector valued function, each on the in t-interval I, and if t_0 is a point in I, then each initial value problem

$$x'(t) = A(t)\,x(t) + f(t)$$
$$x(t_0) = x_0$$

has a unique solution x(t), defined for all t in I. Use this theorem to explain (i.e. prove) why the solution space to the homogeneous system

$$x'(t) = A(t)\,x(t)$$

is n-dimensional.

(10 points)

See the book's discussion on pages 292-293, or page 2 of our class notes, feb24.pdf....basically you use the existence and uniqueness theorem to find a basis of solutions $\{z_j(t)\}$ *by letting* $z_j(t)$ *solve the IVP for the homogeneous system with initial vector* e_j *. (And then you use linearity, i.e. superposition and the uniqueness theorem to prove this collection spans the solutions space (because linear combinations solve every IVP), and then easily verify linear independence. This is a nice theorem for our course because it makes us recall key ideas from Math 2270.*