Name..
I.D. number..

Math 2280-1
Practice Final Exam

April 2006

The exam is closed-book and closed-note. You may use a scientific calculator, but not one which does linear algebra, differential equations, or symbolic integration computations. You will be provided with a Laplace transform table and any non-standard integrals you need. This exam counts for 30% of your final grade, and is written so that there are 150 points possible. Point values are indicated to the right of each problem. Good Luck!!
(NOTE: AS USUAL THIS IS ONLY A SAMPLE OF THE KINDS OF QUESTIONS THAT MAY BE ASKED. IT IS NOT INTENDED TO BE INCLUSIVE OF ALL POSSIBILITIES!)

1) A motorboat weighs 32000 pounds and its motor provides a thrust force of 5000 pounds. Assume the water provides a linear drag, with resistance force equal to 100 pounds for each foot per second of the speed v of the boat.
1a) For the description above, derive the first order differential equation

$$1000\,\frac{dv}{dt} = 5000 - 100\,v$$

for the velocity of the boat at time t.

(3 points)

1b) If the boat started from rest predict its limiting speed, based on the an analysis of the equilibrium solutions to the differential equation above?

(2 points)

1c) Find the explicit solution to the initial value problem for this differential equation, with v(0)=0. Their are FOUR ways you can do this problem (first order linear, separable, general linear, Laplace transform). Solve it each way! (5 points per method).

(20 points)

2) Consider the undamped forced oscillator problem

$$\frac{d^2 x}{dt^2} + 9\,x = 2\cos(\omega\,t)$$

2a) For what value of (positive) omega do you expect resonance?

(3 points)

2b) Find the general solution to this differential equation, in the case when there is no resonance.

(12 points)

3) Consider the following two-tank configuration: In tank one there is uniformly mixed volume of V1 gallons, and pounds of solute x(t). In tank two there is mixed volume of V2 gallons and pounds of solute y(t). Water us pumped into tank one at a constant rate of ri gallons/minute from an outside source, and this water has a constant solute concentration of ci pounds/gallon. Water is pumped from tank one to tank two at constant rate of r1 gallons/minute, from tank two to tank one at constant rate r2 gallons/minute, and out of the tank system (from tank 2) at constant rate ro gallons/minute.

3a) Write the system of first order differential equations which governs the process described above. Do not try to solve these DE's.

(3 points)

3b) Suppose that ri=ro=0, so that no water is flowing into or out of the system. Suppose further that r1=r2, so that V1 and V2 are constant. Set V1=100 gallons, V2=50 gallons, and let r1=r2=100 gallons per hour. Show that in this case the general system your derived in 3a) reduces to the first order system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(2 points)

3c) Use Laplace transform to solve the initial value problem for this tank system:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(10 points)

4) Consider a configuration of two walls, 3 masses and four springs in a linear array, see Figure 5.3.1 page 315. The spring constants, from the left, are k_1, k_2, k_3, k_4. The mass constants, also from the left are m_1, m_2, m_3. As usual, we measure the displacements of the masses from equilibrium by x_1, x_2, x_3, and we choose the positive direction to be to the right.

4a) Find choices of the spring constants and the masses so that the second order system satisfied by the displacements is given by

$$\begin{bmatrix} \frac{d^2 x1}{dt^2} \\ \frac{d^2 x2}{dt^2} \\ \frac{d^2 x3}{dt^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

(3 points)

4b) Find the general solution to the system above. Hint: Use the information that the eigenvectors of the matrix A above are given by

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> eigenvects(A);
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$$[-2, 1, \{[-1, 0, 1]\}], [-2+\sqrt{2}, 1, \{[1, \sqrt{2}, 1]\}], [-2-\sqrt{2}, 1, \{[1, -\sqrt{2}, 1]\}]$$

(6 points)

4c) Describe the three fundamental modes of this system. Which vibrates the quickest, and which vibrates the slowest?

(6 points)

5) Consider the system of differential equations below which models two populations x(t), y(t):

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -x^2 - x\,y + 5\,x \\ -y^2 + x\,y + 3\,y \end{bmatrix}$$

5a) Maybe we should call this system the ‘‘omnivorous predator-prey’’ model, since one of the populations seems to deplete the other one but not be totally reliant on it for survival. Perhaps x(t) and y(t) are measuring how many thousands of each animal are present at time t years. Find all four equilibrium solutions which exist for this system of differential equations.

(8 points)

5b) Classify the stability and type of each of the four equilibrium solutions from part (5a).

(16 points)

5c) Make a rough sketch of the phase field in the first quadrant which is consistent with your information from part (5b). (So, for example, if you are sketching saddle points you needn't worry about exactly what the eigendirections are.) Assuming that both populations start initially with positive values, deduce the possible limiting populations as time aproaches infinity.

(6 points)

6a) Let f be a 2L-periodic function. Write down the Fourier series for f, and write down the formulas for the Fourier coefficients as well.

(8 points)

6b) Let f(x) be the period 2 function obtained by taking the odd extension of the function which equals x(1-x) on the interval [0,1]. Derive the Fourier series for f:

(7 points)

$$f(x) = \sum_{n=odd} \left(8\,\frac{\sin(n\,\pi\,x)}{n^3\,\pi^3} \right)$$

You may wish to use the integration formulas

$$\int x\,\sin(n\,\pi\,x)\,dx = \frac{\sin(n\,\pi\,x) - n\,\pi\,x\cos(n\,\pi\,x)}{n^2\,\pi^2}$$

$$\int x^2\,\sin(n\,\pi\,x)\,dx = \frac{-n^2\,\pi^2\,x^2\cos(n\,\pi\,x) + 2\cos(n\,\pi\,x) + 2\,n\,\pi\,x\sin(n\,\pi\,x)}{n^3\,\pi^3}$$

7a) Derive all possible product solutions u(x,t)=X(x)T(t) to the heat equation

$$u_t = k\,u_{xx}$$

with the ‘‘fixed boundary temperature’’ assumption that u(0)=u(L)=0.

(7 points)

7b) Solve the initial boundary-value problem for the heat equation, where the initial temperature of a rod on the interval 0<x<1 is given by f(x)=x(1-x), and where the endpoint temperatures are held at temperature zero for positive time values. Write the solution for general heat diffusivity k. (Hint: the sine series for f was given in a previous problem.)

(8 points)

8) Consider the wave equation

$$u_{tt}(x, t) = 9\, u_{xx}(x, t)$$

on the interval 0<x<1, for positive time, and consider the initial boundary value problem where the endpoints are fixed (u(0,t)=u(1,t)=0), where the initial profile is given by the function $u(x, 0) = f(x) = \sin(\pi x)$, and where the initial velocity is $u_t(x, 0) = 0$. Find the solution $u(x, t)$.

(10 points)

9) The existence and uniqueness theorem for linear first order systems of differential equations says that if A(t) is a continuous, n by n matrix-valued function of t, and f(t) is a continuous n-vector valued function, each on the in t-interval I, and if t_0 is a point in I, then each initial value problem

$$x'(t) = A(t)\, x(t) + f(t)$$
$$x(t_0) = x_0$$

has a unique solution x(t), defined for all t in I. Use this theorem to explain (i.e. prove) why the solution space to the homogeneous system

$$x'(t) = A(t)\, x(t)$$

is n-dimensional.

(10 points)