Math 2280-1
Fri March 31
Exam Tuesday! §3.5-6.4

HW for Fri 4/7
7.1 ③ 7 (8, 13, 20) 21, (23, 28)
7.2 3, ④ 5 (6, 14)

Chapter 7 Laplace transform.
↓
one of several linear transformations in the field of DE's & PDE's, which have the amazing property of transforming linear DE's into algebraic problems.

for $f:\underset{t}{[0,\infty)}\to\mathbb{R}$

$$\boxed{\underset{\overset{\shortparallel}{F(s)}}{\mathcal{L}\{f(t)\}(s)} := \int_0^\infty e^{-st} f(t)\,dt}$$

(it is assumed $|f(t)| \le Ce^{Mt}$ for t large, so that the F(s) integral converges for $s > M$ (or $\mathrm{Re}(s) > M$)

Easy fact: $\mathcal{L}$ is linear ( $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = c_1 F_1(s) + c_2 F_2(s)$ )
Hard fact: $\mathcal{L}$ is 1-1, so has an inverse defined on its image subspace
Easy fact: inverses of linear transformations are always linear too
(just take $\mathcal{L}^{-1}$ of both sides of)

Examples

$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\,1\,dt = \frac{e^{-st}}{-s}\bigg]_0^\infty = \frac{1}{s} \qquad (\mathrm{Re}\, s > 0)$$

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{t(a-s)}\,dt = \frac{1}{a-s} e^{t(a-s)}\bigg]_{t=0}^{\infty} = \frac{1}{s-a} \qquad (\mathrm{Re}\, s > a)$$

$$\cos kt + i \sin kt = e^{ikt} \qquad (k \text{ real})$$

$$\Rightarrow \mathcal{L}\{\cos kt\}(s) + i\,\mathcal{L}\{\sin kt\}(s) = \mathcal{L}\{e^{ikt}\} = \frac{1}{s-ik} \qquad (a = ik)$$

$$= \frac{s+ik}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$$

Most important:

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv} = e^{-st} f(t)\bigg]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt = -f(0) + sF(s)$$

$$\text{so, } \mathcal{L}\{f''(t)\}(s) = -f'(0) + s\,\underbrace{\mathcal{L}\{f'(t)\}(s)}_{-f(0)+sF(s)} = s^2 F(s) - s f(0) - f'(0) \quad \text{etc.}$$

## Laplace transform magic:

**Example 1**

Solve

$$\begin{cases} x''(t) + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

**Method 1**: Chapter 3! on upcoming exam!

this is undamped forced oscillation at unnatural frequency!

$x_H$

$x_p$

$x_p + x_H$, match I.C.'s.

**Method 2**:

$$x'' + 4x = 10 \cos 3t$$

iff

$$\mathcal{L}\{x''(t) + 4x(t)\}(s) = \mathcal{L}\{10\cos 3t\}(s)$$

i.e.

$$s^2X(s) - s\underset{\overset{\uparrow}{2}}{x(0)} - \underset{\overset{\uparrow}{1}}{x'(0)} + 4X(s) = 10\frac{s}{s^2+9}$$

$$X(s)(s^2+4) = 10\frac{s}{s^2+9} + 2s + 1$$

$$X(s) = 10\frac{s}{(s^2+9)(s^2+4)} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

done in Laplace land!
now use partial fracs & table to invert Laplace $X(s)$ to get $x(t)$

$$\frac{1}{(s^2+9)(s^2+4)} = \frac{1}{5}\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)$$

So

$$X(s) = \frac{2s}{s^2+4} - \frac{2s}{s^2+9} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$X(s) = \frac{4s}{s^2+4} + \frac{1}{s^2+4} - \frac{2s}{s^2+9}$$

$$\boxed{x(t) = 4\cos 2t + \tfrac{1}{2}\sin 2t - 2\cos 3t}$$

| $f(t)$ | $F(s)$ | |
|---|---|---|
| $1$ | $1/s$ | $\operatorname{Re} s > 0$ |
| $e^{at}$ | $1/(s-a)$ | $\operatorname{Re} s > a$ |
| $\cos kt$ | $s/(s^2+k^2)$ | |
| $\sin kt$ | $\frac{k}{s^2+k^2}$ | |
| $f'(t)$ | $sF(s) - f(0)$ | |
| $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ | |
| $f'''(t)$ | $s^3F(s) - s^2f(0) - sf'(0) - f''(0)$ | |
| $\cosh kt$ | $s/(s^2-k^2)$ | |
| $\sinh kt$ | $k/(s^2-k^2)$ | |
| $\vdots$ | | |

(see book cover!)

| | |
|---|---|
| $e^{at}\cos kt$ | $\frac{s-a}{(s-a)^2+k^2}$ |
| $e^{at}\sin kt$ | $\frac{k}{(s-a)^2+k^2}$ |
| $\vdots$ | |

Use

$$\mathcal{L}\{e^{(a+ki)t}\}(s) = \frac{1}{s-(a+ki)}$$

and take real & imag parts.

Review sheet for exam 2 (which is on Tuesday)

To cover

$3.5-3.6$, $4.1$, $5.1-5.6$, $6.1-6.4$

↑ not 3.7 ↑ not 4.2, 4.3 ↑ not 6.5

≈25%
- Chapter 6: nonlinear systems of DE's
  - equilibria for autonomous systems
  - stability & asymptotic stability defs.
  - stability criteria (~~based on~~ for linear & nonlinear problems
    - linearization
  - geometry of different equilibria for 2 first order DE systems
  - §6.3, §6.4 applications (interacting species, mechanical systems)
  - reconstructing global phase portraits from linearization info at equilibria.

≈40%
- Chapter 5: linear systems of DE's
  - eval-evec method for solving $\vec{x}' = A\vec{x}$ (& tanks)
  - related method for solving $\vec{x}'' = A\vec{x}$ (for undamped spring systems)
    - $\vec{x}_p$ for $\vec{x}'' = A\vec{x} + \vec{b}\cos\omega t$ (forced spring systems)
    - $\vec{x} = \vec{x}_p + \vec{x}_H$
    - IVP
  - FMS, $e^{At}$, variation of parameters for $\vec{x}' = A\vec{x} + \vec{f}(t)$
    - (also undet'd coeffs)
  - what to do about defective eigenvalues

≈10%
- Chapter 4: §4.1 Theory for 1st order systems.
  - ∃! for 1st order IVP
  - converting nth order DE's (or systems) to first order systems.
  - dim of soln space to 1st order homogeneous linear systems of DE & why

≈25%
- Chapter 3.5-3.6
  - undet'd coeffs
  - forced oscillations for springs
    - undamped ~ beating, resonance
    - damped ~ practical resonance