Chapter 7 Laplace transform.

One of several linear transformations in the field of DE's & PDE's, which have the amazing property of transforming linear DE's into algebraic problems.

For \( f: (0, \infty) \rightarrow \mathbb{R} \):

\[
\mathcal{L} \{ f(t) \}(s) = \int_0^\infty e^{-st} f(t) \, dt
\]

(it is assumed \( |f(t)| \leq Ce^{Mt} \) for \( t \) large, so that the \( F(s) \) integral converges for \( s > M \) (or \( \text{Re}(s) > M) \).

**Easy fact:** \( \mathcal{L} \) is linear. \( \mathcal{L} \{ c_1 f_1(t) + c_2 f_2(t) \}(s) = c_1 F_1(s) + c_2 F_2(s) \).

**Hard fact:** \( \mathcal{L} \) is \( L^1 \), so has an inverse defined on its image subspace.

**Easy fact:** inverses of linear transformations are always linear too (just take \( \mathcal{L}^{-1} \) of both sides).

**Examples**

\[
\mathcal{L} \{ 1 \}(s) = \int_0^\infty e^{-st} \, dt = \frac{e^{-st}}{-s} \bigg|_0^\infty = \frac{1}{s} \quad \text{(Re} s > 0\text{)}
\]

\[
\mathcal{L} \{ e^{at} \}(s) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{(a-s)t} \, dt = \frac{1}{a-s} e^{(a-s)0} = \frac{1}{s-a} \quad \text{(Re} s > a\text{)}
\]

\[\cos kt + i \sin kt = e^{ikt} \quad \text{(k \text{ real})}
\]

\[\Rightarrow \mathcal{L} \{ \cos kt \}(s) + i \mathcal{L} \{ \sin kt \}(s) = \mathcal{L} \{ e^{ikt} \} = \frac{1}{s-ik} = \frac{s+ik}{s^2+k^2}
\]

\[\Rightarrow \mathcal{L} \{ \cos kt \}(s) = \frac{s}{s^2+k^2}
\]

\[\mathcal{L} \{ \sin kt \}(s) = \frac{k}{s^2+k^2}
\]

**Most important:**

\[
\mathcal{L} \{ f'(t) \}(s) = \int_0^\infty e^{-st} f'(t) \, dt = e^{st} f(t) \bigg|_0^\infty - \int_0^\infty se^{-st} f(t) \, dt = -f'(0) + sF(s)
\]

So, \( \mathcal{L} \{ f''(t) \}(s) = -f''(0) + s \mathcal{L} \{ f'(t) \}(s) = s^2F(s) - sf(0) - f'(0) \) etc.
Example 1

Solve

\[
\begin{align*}
x''(t) + 4x(t) &= 10 \cos 3t \\
x(0) &= 2 \\
x'(0) &= 1
\end{align*}
\]

Method 1: Chapter 5! an upcoming exam!

This is an underdamped forced oscillation at an unnatural frequency!

\[x_H, \quad x_p, \quad x_H + x_p, \text{ match I.C.'s.}\]

Method 2:

\[x'' + 4x = 10 \cos 3t \]

\[
\mathcal{L}\{x''(t) + 4x(t)\} = \mathcal{L}\{10 \cos 3t\}
\]

\[
i.e. \quad s^2X(s) - sx(0) - x'(0) + 4X(s) = 10 \frac{s}{s^2 + 9}
\]

\[
\uparrow \quad \uparrow 
\]

\[
X(s) = \frac{10s}{s^2 + 9} + 2s + 1
\]

\[
X(s) = \frac{10s}{(s^2 + 9)(s^2 + 4)} + \frac{2s}{s^2 + 4} + \frac{1}{s^2 + 4}
\]

\[
\text{done in Laplace land!}
\]

\[
\text{now use partial fracs & table to invert}
\]

\[
\text{Laplace } X(s) \text{ to get } x(t)
\]

\[
\frac{1}{(s^2 + 9)(s^2 + 4)} = \frac{1}{5} \left( \frac{1}{s^2 + 9} - \frac{1}{s^2 + 4} \right)
\]

\[
\left. \begin{array}{c}
\mathcal{L}\{1\} = \frac{1}{s} \\
\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \\
\mathcal{L}\{\cosh kt\} = \frac{s}{s^2 + k^2} \\
\mathcal{L}\{\sinh kt\} = \frac{k}{s^2 + k^2} \\
\mathcal{L}\{e^{akt}\} = \frac{1}{s-a} \\
\mathcal{L}\{f(t) - f(0)\} = sF(s) - f(0) \\
\mathcal{L}\{f''(t) - f(0)\} = s^2F(s) - sf(0) - f'(0) \\
\mathcal{L}\{f'''(t) - f(0)\} = s^3F(s) - sf(0) - f'(0) - f''(0)
\end{array} \right\}
\]

\[
\text{Use}
\]

\[
\mathcal{L}\{e^{(a+bi)t}\} = \frac{1}{s-(a+bi)}
\]

\[
\text{and take real & imaginary parts.}
\]

\[
x(t) = 4 \cos 2t + \frac{1}{2} \sin 2t - 2 \cos 3t
\]
Review sheet for exam 2 (which is on Tuesday)

To cover:
3.5-36, 4.1, 5.1-5.6, 6.1-6.4

• Chapter 6: nonlinear systems of DE’s
  - equilibrium for autonomous systems
  - stability & asymptotic stability of DE’s
  - stability in terms for linear & nonlinear problems
  - linearization
  - geometry of different equilibria for 2 first order DE systems
  - 4.6.3, 4.6.4 applications (interacting species, mechanical systems)
  - reconstructing global phase portraits from linearization info at equilibria

• Chapter 5: linear systems of DE’s
  - evaluate method for solving \( x' = Ax \) (2 tanks)
  - related method for solving \( x'' = Ax \) (for undamped spring systems)
  - \( x_p \) for \( x'' = Ax + f(t) \) (forced spring systems)
  - evaluate method for solving \( x = x_p + x_H \)
  - IVPS, \( e^{At} \), variation of parameters for \( x' = Ax + f(t) \)
  - (also undet'd coeffs)
  - what to do about defective eigenvalues

• Chapter 4: 5.4.1 Theory for 1st order systems
  - \( x' \) for 1st order IVPS
  - converting an order DE’s (or systems) to first order systems
  - linear change of scalars space to 1st order homogeneous systems of DE & why

• Chapter 3.5-3.6
  - undet’d coeffs
  - forced oscillations in springs
  - undamped ~ beating, resonance
damped ~ practical resonance