Math 2280-1
Fri 3/24

§6.2.

Equilibria, stability & phase portraits for nonlinear systems of DE's.

HW for Fri 3/31

6.1 5, 8, 11, 15, 20, 24
6.2 5, 6, 7, 8, 9, 14, 15, 19, 27, 30
6.3 8, 9, 10, 14, 15, 16, 17
6.4 12, 13, 14, 15, 16

I recommend <u>pplane</u> when the text wants you to produce computer pictures/checks

- Prove the theorem on page 4 Wed about the stability of $\vec{x}_* = \vec{0}$ for

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

(there's room on page 5 Wed.)

- Discuss the classification of $\vec{x}_* = \vec{0}$ for

$$\vec{x}\,' = A\vec{x} \quad (n=2)$$

and the corresponding table for equilibria of non-linear systems, based on "linearization"
↑
page 3 today.

Eigenvalues of A	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node
Complex conjugate	Spiral point
Pure imaginary	Center

FIGURE 6.2.9. Classification of the critical point (0, 0) of the two-dimensional system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

linear

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

FIGURE 6.2.12. Classification of critical points of an almost linear system.

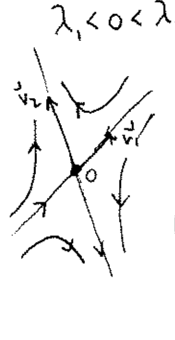
nonlinear – dominated by linearization

If λ real: $\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

or (if λ_1 is defective)

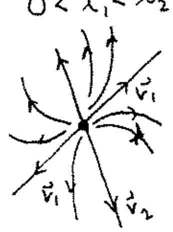
$$\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 \left(e^{\lambda_1 t}\right)\left(t\vec{v}_1 + \vec{u}\right)$$

$\lambda_1 < 0 < \lambda_2$



$\boxed{\text{saddle}}$
(unstable)

$0 < \lambda_1 < \lambda_2$



(unstable)
improper node
(or <u>nodal source</u>)

$\lambda_1 = \lambda_2$
"<u>proper</u>"

$\lambda_2 < \lambda_1 < 0$
stable improper node
(<u>nodal sink</u>)

$\lambda_1 = \lambda_2$
proper

If $\lambda = a + bi$ complex:

$a < 0$ (asy. stable) spiral sink

$a > 0$ (unstable) spiral source

$a = 0$ (stable) center for linear; indeterminant for non-linear

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

<u>Complex eigenvalues details for page 1 classification</u>

$$\lambda = a \pm bi \qquad \vec{w} = \vec{u} \pm i\vec{v}$$

$$A(\vec{u} + i\vec{v}) = (a+bi)(\vec{u} + i\vec{v})$$
$$A(\vec{u} - i\vec{v}) = (a-bi)(\vec{u} - i\vec{v})$$

$$\Rightarrow \begin{aligned} A\vec{u} &= a\vec{u} - b\vec{v} \\ A\vec{v} &= b\vec{u} + a\vec{v} \end{aligned}$$

(add eqtns, divide by 2)
(subt eqtns, divide by $2i$)

$$\Rightarrow [A]_{\{\vec{u},\vec{v}\}} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} := B$$

(a rotation-dilation matrix!)

$$S^{-1} A S = B \qquad S = [\vec{u} | \vec{v}].$$

$$\vec{x}' = A\vec{x}$$
$$\vec{x}' = SBS^{-1}\vec{x}$$

$$(S^{-1}\vec{x})' = B(S^{-1}\vec{x}) \quad ; \quad \text{write } \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_{\{\vec{u},\vec{v}\}}}_{=\,S^{-1}\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} z \\ w \end{bmatrix}$$

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

sol'n! $$\begin{bmatrix} z \\ w \end{bmatrix} = C_1 e^{at} \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} + C_2 e^{at} \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}$$

$$\begin{bmatrix} z \\ w \end{bmatrix} = e^{at} \begin{bmatrix} C_1 \cos bt + C_2 \sin bt \\ C_1 \cos(bt + \pi/2) + C_2 \sin(bt + \pi/2) \end{bmatrix}$$

$$= e^{at} \begin{bmatrix} C \cos(bt - \alpha) \\ C \cos(bt + \pi/2 - \alpha) \end{bmatrix}$$

$$\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = C e^{at} \begin{bmatrix} \cos(bt - \alpha) \\ -\sin(bt - \alpha) \end{bmatrix}$$

circle radius C

spirals outward if $a > 0$
inward if $a < 0$

Finally, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ is an ellipse if $a = 0$ else an elliptical spiral

Linearization (works for systems of n DE's; illustrated for $n=2$)

Let
$$(1)\quad \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$F(x_*, y_*) = F(P) = 0$$
$$G(x_*, y_*) = G(P) = 0$$

write
$$x(t) = x_* + u(t)$$
$$y(t) = y_* + v(t)$$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\frac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v) \to (0,0)$

$$x' = F(x_*+u, y_*+v) = F(\overset{0}{\cancel{x_*, y_*}}) + F_x(x_*, y_*)u + F_y(x_*, y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u, y_*+v) = G(\underset{0}{\cancel{x_*, y_*}}) + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

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affine approx.

$$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$$
$$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$$

where the partial derivs of F & G are evaluated at the equil. pt.

$$(2)\qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{“}A\text{”}} \begin{bmatrix} u \\ v \end{bmatrix}$$

this is the linearization of (1), at (x^*, y^*).
the eigenvector data of A determines stability for the nonlinear system (1), in the non borderline cases.

the matrix A is called the <u>Jacobian matrix</u> for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

Return to Wed example:

$$x' = x - y - x^2 + xy$$
$$y' = -y - x^2$$

equilibria were

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Find the Jacobian matrices at these three equilibria (we really already did $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$), and verify that the phase portrait agrees with the classification scheme on page 1

$$\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 1-2x+y & -1+x \\ -2x & -1 \end{bmatrix}$$

@ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $J = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ $\quad |J - \lambda I| = (\lambda-1)(\lambda+1)$: saddle.

@ $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$:

@ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$: