Math 2280-1
Wed 22 March

§6.1-6.2
autonomous systems of (1st order) DE's

(1) $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$

usually $n=2$, so (1) reads

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

- equilibrium soltn
- stable
  unstable
  asymptotically stable

Math 2280-1
Wednesday March 22
Linearization and stability, sections 6.1-6.2

We'll work with a fresh example (because it has lots of nice subexamples). This is the system of section 6.1 homework, #8:

$$\frac{d}{dt}x(t) = x(t) - y(t) - x(t)^2 + x(t)\,y(t) \quad = \quad x - y - x^2 + xy$$
$$\frac{d}{dt}y(t) = -y(t) - x(t)^2 \quad = \quad -y - x^2$$

Find the equilibrium solutions:

Use the figure on the next page to guess about the stability properties of each equilibrium solution you found:

```
> restart:
> with(plots):with(linalg):with(DEtools):
> deqtn:={diff(x(t),t)=x(t)-y(t)-x(t)^2+x(t)*y(t),
          diff(y(t),t)=-y(t)-x(t)^2};
```

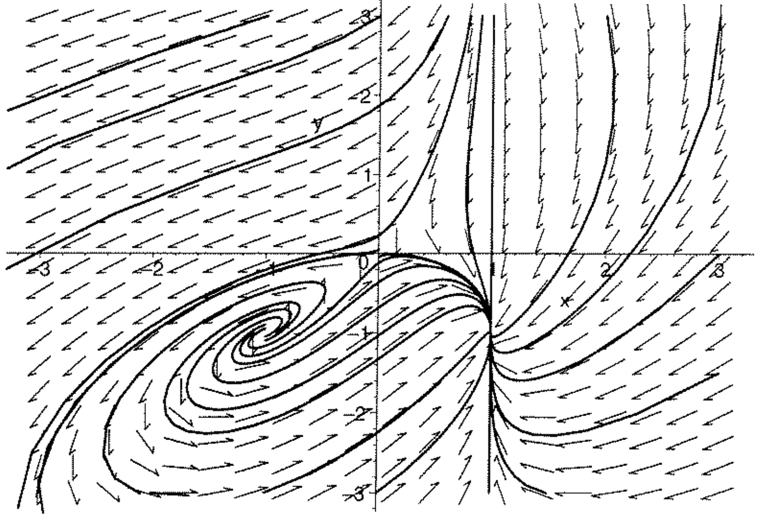
$$deqtn := \{\frac{d}{dt}\,x(t) = x(t) - y(t) - x(t)^2 + x(t)\,y(t), \frac{d}{dt}\,y(t) = -y(t) - x(t)^2\}$$

```
> ICs:=[[x(0)=-1,y(0)=3],[x(0)=0,y(0)=3],
  [x(0)=.9,y(0)=3],
  [x(0)=.6,y(0)=3],[x(0)=.8,y(0)=3],
  [x(0)=1,y(0)=3],[x(0)=2,y(0)=3],
  [x(0)=3,y(0)=3],[x(0)=3,y(0)=0],
  [x(0)=3,y(0)=-1.5],[x(0)=0.05,y(0)=0],
  [x(0)=-.05,y(0)=0],[x(0)=0,y(0)=-.05],
  [x(0)=0,y(0)=.05],
  [x(0)=1.5,y(0)=-3],
  [x(0)=1,y(0)=-3],[x(0)=0,y(0)=-3],
  [x(0)=-1,y(0)=-3],[x(0)=-2,y(0)=-3],
  [x(0)=-1.1,y(0)=-1],[x(0)=-1,y(0)=-.9],
  [x(0)=-.9,y(0)=-1],[x(0)=-1,y(0)=-1.1],
  [x(0)=-1,y(0)=-.95],[x(0)=-1,y(0)=-1.05]]:
  #initial conditions to make a picture like Figure 6.1.16
> phaseportrait(deqtn,[x(t),y(t)],t=0..15,ICs,x=-3..3,y=-3..3,arrows
  =small,
  stepsize=.1,color=black,linecolor=black,title=`Figure 6.1.16`);
```

$x' = x - y - x^2 + xy$

$y' = -y - x^2$

Figure 6.1.16



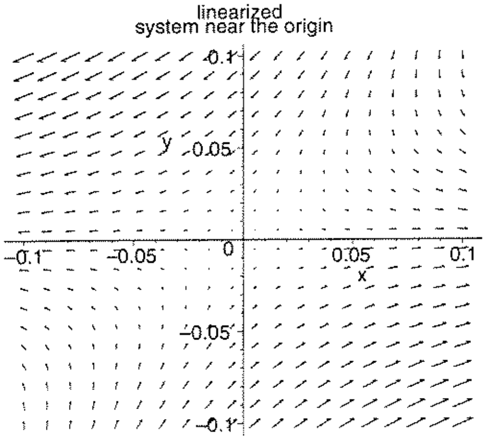
Near the origin equilibrium, our system is very close to the system

$$\begin{bmatrix} \frac{d}{dt}\mathrm{x}(t) \\ \frac{d}{dt}\mathrm{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Why?

And, if we plot the phase portrait for this system, (actually I prefer the fieldplot, which doesn't normalize the tangent vector field), it looks very close to a magnification of the phase portrait near the origin equilibrium:

```
> fieldplot([x-y,-y],x=-.1..(.1),y=-.1..(.1),color=black,title='line
  arized
  system near the origin');
```



Explain what the picture above has to do with the following data, by finding the general solution to the first order system and explaining the geometry of the trajectories.

```
> A:=matrix(2,2,[1,-1,0,-1]);
```

$$A := \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

```
> eigenvectors(A);
```

```
[-1, 1, {[1, 2]}], [1, 1, {[1, 0]}]
```

Assuming that (on small scales) the stability of the linearized systems governs that of the non-linear system, what does your eigenvalue, eigenvector analysis tell you about stability (for both systems), near the origin equilibrium?

We will carry out this linearization procedure at the other two equilibria on Friday. We will use this general theorem:

Theorem Consider the linear system of DE's

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

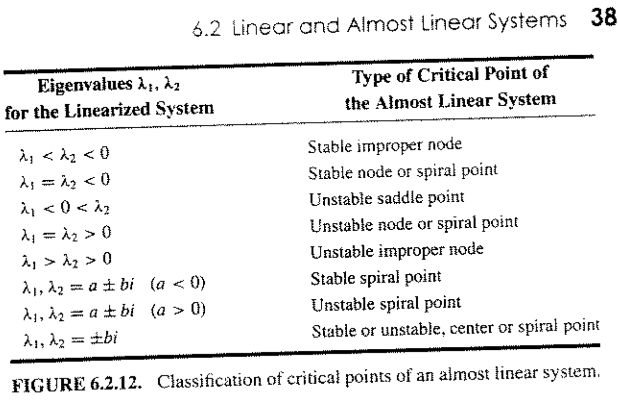
and the equilibrium solution $\vec{x}_* = \vec{0}$.

- If each eigenvalue λ of A satisfies
  $$\mathrm{Re}(\lambda) < 0 \qquad (\text{the real part of } \lambda \text{ is negative, i.e. } \lambda = a+bi \text{ with } a<0)$$
  Then $\vec{x}_* = \vec{0}$ is asymptotically stable
- If at least one eigenvalue λ of A satisfies $\mathrm{Re}(\lambda) > 0$, then $\vec{x}_* = \vec{0}$ is unstable

(If you replace the strict ineqs $<, >$ with $\leq, \geq$, you must consider subcases).

prove this theorem!

In the case $n=2$ there is a short list of the geometric structure of the phase portrait near an equilibrium sol'n, which we will discuss more on Friday. Using this table in the book, give names to the 3 critical pts (equilibria) in our main example:



6.2 Linear and Almost Linear Systems **387**

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

FIGURE 6.2.12. Classification of critical points of an almost linear system.