Math 2280-1

Tues March 21

Chapter 6: Nonlinear systems of DE's

HW (part of next weeks' assignment)

6.1 5, 8, 11, 15, 20, 24

§6.1 . Phase plane

(nonlinear) system of two 1st order DE's

(1) $$\frac{dx}{dt} = F(x,y,t)$$
$$\frac{dy}{dt} = G(x,y,t)$$

example (§6.3)

$x(t)$ = prey population (fish, rabbits, etc.)

$y(t)$ = predator population (sharks, foxes, etc)

$$\begin{cases} \frac{dx}{dt} = ax - pxy \\ \frac{dy}{dt} = -by + qxy \end{cases}$$

($-cx^2$, if you want the prey to be logistic).

understand model assumptions:

example (§6.4)

$x(t)$ = pendulum angle $\theta(t)$

$y(t) = x'(t)$ = angular velocity $\theta'(t)$

$$\begin{cases} x' = y \\ y' = -\frac{g}{L}\sin x \end{cases}$$

Def: If the only dependendence of F and G on t is through $x(t)$ & $y(t)$, then the system (1) is called autonomous, i.e.

(2) $$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

In this case we call x-y plane the phase plane, and the solution curves $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ are called trajectories.

~~They follow the tangent vector field~~ $\begin{bmatrix} F \\ G \end{bmatrix}$

constant solutions to (2) are called equilibrium solutions

They are exactly the solutions to the (non)linear system

$$(3) \qquad \begin{aligned} 0 &= F(x,y) \\ 0 &= G(x,y) \end{aligned}$$

example : competing species; say $x(t)$ = rabbit population, $y(t)$ = squirrel population

perhaps

$$\frac{dx}{dt} = 14x - 2x^2 - xy$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} \underbrace{- xy}_{\text{competition}}$$

Find the equilibrium sol'ns

ans $(0,0)$
$(0,8)$
$(7,0)$
$(4,6)$

It will be important to know whether equilibrium sol'ns are stable or unstable

Def $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is a stable equilibrium for (2) if it is a constant sol'tn (equilibrium) (i.e. satisfies (3)),
" $\vec{x}^*$ and if $\forall\, \varepsilon > 0\ \exists\ \delta > 0$ s.t.
whenever
$\|\vec{x}^* - \vec{x}_0\| < \delta$ $\qquad \left(\|\vec{x}^* - \vec{x}_0\| = \sqrt{(x_0 - x^*)^2 + (y_0 - y^*)^2} \right)$
then the sol'n to (2) with
$\vec{x}(0) = \vec{x}_0$ satisfies $\|\vec{x}(t) - \vec{x}^*\| < \varepsilon \quad \forall\ t > 0$.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is unstable equilibrium if it is an equilibrium sol'n which is not stable.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is asymptotically stable iff it is stable and $\exists\ \delta > 0$ s.t.
$\|\vec{x}^* - \vec{x}_0\| < \delta \implies$ the IVP sol'tn with $\vec{x}_0 = \vec{x}(0)$
satisfies $\lim_{t \to \infty} \vec{x}(t) = \vec{x}^*$

Here's the phase portrait for the rabbit-squirrel model.
Guess the stability of the 4 equilibrium sol'ns:
(And discuss any predictions you might have for rabbit-squirrel populations

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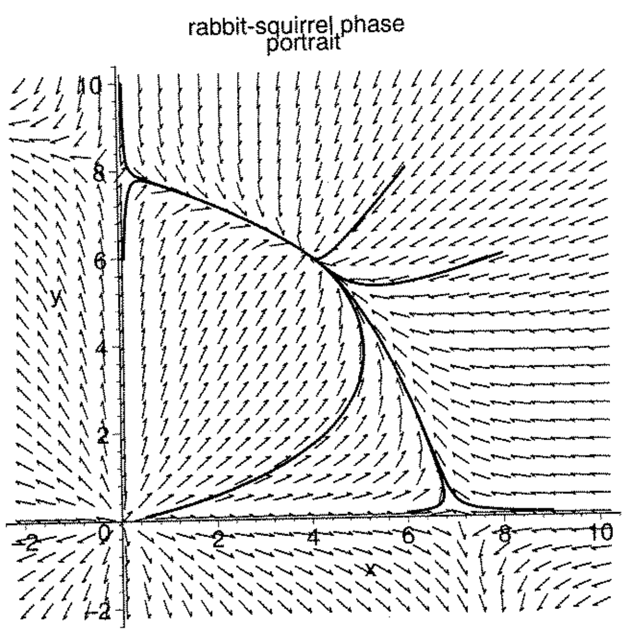
>
> with(DEtools):
> phaseportrait([diff(x(t),t)=14*x(t)-2*x(t)^2-x(t)*y(t),
  diff(y(t),t)=16*y(t)-2*y(t)^2-x(t)*y(t)],
  [x(t),y(t)],t=0..2,[[x(0)=.5,y(0)=.1],[x(0)=.1,y(0)=10],[x(0)=.1,y
  (0)=6],
  [x(0)=6,y(0)=.1],[x(0)=9,y(0)=.1],
  [x(0)=8,y(0)=6],[x(0)=6,y(0)=8]],stepsize=.01,x=-2..10,y=-2..10,
  linecolor=black,color=black,dirgrid=[30,30],title=`rabbit-squirrel
  phase
  portrait`);
```

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We will understand stability by <u>linearizing</u> near equilibrium sol'tns.

<u>Example</u>: linearize rabbit-squirrel model near $\begin{bmatrix}4\\6\end{bmatrix}$

Let $x = 4+u$, $y = 6+v$ with $|u|, |v|$ small

Then
$$\frac{du}{dt} = \frac{dx}{dt} = 14(4+u) - 2\underbrace{(4+u)^2}_{(16+8u+u^2)} - (4+u)(6+v) =$$
$$= \begin{array}{l}56\\-32\\-24\end{array} \quad \begin{array}{l}+u(14-16-6)\\+v(-4)\end{array} \quad + -4u^2 - uv$$

$$\frac{dv}{dt} = \frac{dy}{dt} = 16(6+v) - 2\underbrace{(6+v)^2}_{(36+12v+v^2)} - (4+u)(6+v)$$
$$= \begin{array}{l}96\\-72\\-24\end{array} \quad \begin{array}{l}+u(-6)\\+v(16-24-4)\end{array} \quad \begin{array}{l}-2v^2\\-uv\end{array}$$

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix} = \underbrace{\begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}}_{\text{linear piece}} + \underbrace{\begin{bmatrix}-4u^2-uv\\ -2v^2-uv\end{bmatrix}}_{\text{error}}$$

error; if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \delta$ then $\|\text{error}\| \le \delta^2 (8)$ is tiny.

linearization $\begin{bmatrix}u'\\v'\end{bmatrix} = \begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$

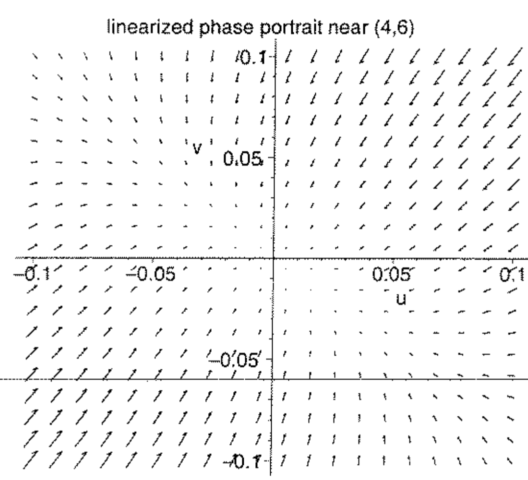
```
> with(linalg):with(plots):
> Digits:=4:
> A:=matrix(2,2,[-8,-4.0,-6,-12]);
  eigenvects(A);
```

$$A := \begin{bmatrix}-8 & -4.0\\ -6 & -12\end{bmatrix}$$

[-4.708, 1, {[0.7722, -0.6354]}], [-15.29, 1, {[0.4895, 0.8923]}]

```
> fieldplot([-8*u-4*v,-6*u-12*v],u=-.1..(.1),v=-.1..(.1),
  color=black,title=`linearized phase portrait near (4,6)`);
```

compare to nonlinear phase portrait, near $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$, magnified.



You can linearize near any equilibrium sol'tn, for any autonomous system. Then a deep thm says the linearized system's stability governs the non-linear system's stability. So we will need...

<u>Theorem</u>: (today you guess) (then see if you can prove)

Let $[A]_{n\times n}$

Then $\vec{x} = \vec{0}$ is an equilibrium sol'tn for

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

What conditions on eigenvalues of A guarantee <u>asymptotic stability</u>? at $\vec{x}^* = \vec{0}$

What condition(s) guarantee(s) $\vec{x}^* = \vec{0}$ is <u>unstable</u>?