Math 2280-1
Monday 3/20

HW for Fri 3/24

5.4 1, 7, 11, 29
5.5 1, 3, 11, 23, 33, 36
5.6 1, 13, 15, 19, 23

5.4: #1 : work by hand. No direction field necessary
7, 11 work by hand.
29: use MAPLE to find generalized evects & chains

5.5 #1, 3 work by hand except you may use maple for evects
11, 23, 33, 36 by hand (use tech. for evects if relevant)
also have Maple compute matrix exponentials as check

5.6 #13, 15 by hand.
#19, 23 use Maple to compute variation of params sol's & solve IVP's, if desired

Do you recall, for the 1st order system

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

what is a FMS?
what is e^{At} ?

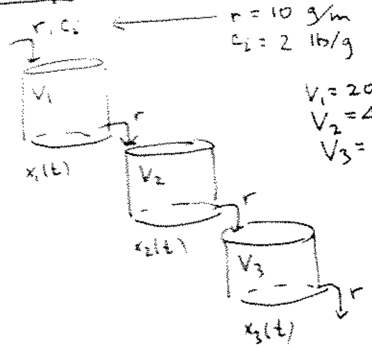
Could you compute these?
Could you use them to solve IVP's?

§5.6 Nonhomogeneous linear systems of DE's :

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \underbrace{\vec{f}(t)}_{\text{nonhomogeneous term}}$$

- undetermined coeff's
- variation of parameters

Example (p. 359)



$r = 10$ g/m
$c_i = 2$ lb/g

$V_1 = 20$ g
$V_2 = 40$
$V_3 = 50$

$\vec{x}(0) = \vec{0}$.

Explain why the 1st order system for $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ is

$$* \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$$

general sol'n is

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

as usual.

eigenvects(A) = [−.5, 1, {[5, −6, 5]}], [−.25, 1, {[0, 1, −5]}], [−.2, 1, {[0, 0, 1]}]   (Maple)

So, for $x_H(t)$ a FMS $\Phi(t)$ is

$$\Phi(t) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

and general sol'tn to $\vec{x}_H(t) = \Phi(t)\vec{c}$.

For $\vec{x}_p(t)$ we guess a constant vector, since

$$L(\vec{x}) := \vec{x}' - A\vec{x}$$

transforms constant vectors to constant vectors   [This is undetermined coeff's easy case for systems!]

If $\vec{x}_p(t) = \vec{k}$, then plug into DE: $\vec{x}' = A\vec{x} + \vec{b}$, page 1

$$\vec{0} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \vec{k} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} \quad ;$$

$$\vec{k} = A^{-1} \begin{bmatrix} -20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} \quad \text{(Maple)}$$

general sol'n

$$\vec{x}(t) = \vec{k} + \Phi(t)\vec{c} \qquad (\vec{x}_p + \vec{x}_H), \quad \text{with } \Phi(t) = \left[e^{-.5t}\begin{bmatrix} 5 \\ -6 \\ 5 \end{bmatrix}, e^{-.25t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}, e^{-.2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

If $\vec{x}(0) = \vec{0}$ (as in example),

$$\vec{0} = \vec{k} + \Phi(0)\vec{c}$$

$$\vec{c} = \Phi(0)^{-1}(-\vec{k}) = \begin{bmatrix} -40/3 \\ -160 \\ -\frac{2500}{3} \end{bmatrix} \quad \text{(Maple)}$$

so $$\vec{x}(t) = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} - \frac{40}{3} e^{-.5t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} - 160\, e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} - \frac{2500}{3} e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

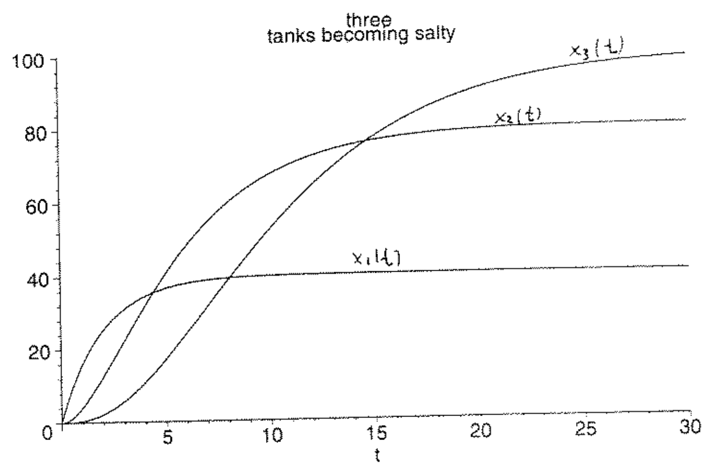
```
> x1:=t->40-40*exp(-.5*t);
  x2:=t->80+80*exp(-.5*t)-160*exp(-.25*t);
  x3:=t->100-200/3*exp(-.5*t)+5*160*exp(-.25*t)-
     2500/3*exp(-.2*t);
```

$$x1 := t \to 40 - 40\, e^{(-0.5\, t)}$$

$$x2 := t \to 80 + 80\, e^{(-0.5\, t)} - 160\, e^{(-0.25\, t)}$$

$$x3 := t \to 100 - \frac{200}{3} e^{(-0.5\, t)} + 800\, e^{(-0.25\, t)} - \frac{2500}{3} e^{(-0.2\, t)}$$

```
> plot({x1(t),x2(t),x3(t)},t=0..30,color=black,title=`three
  tanks becoming salty`);
```




```
>
```

but, for harder nonhomogeneous problems, you'll prefer to use variation of parameters! ☺

Variation of parameters (p. 361)

$$\vec{x}'(t) = P(t)\vec{x} + \vec{f}(t)$$

Let $\Phi(t)$ be a FMS for $\vec{x}'(t) = P(t)\vec{x}$, the homogeneous system.

Try for $\vec{x}_p(t) = \Phi(t)\vec{u}(t)$

$$\Leftrightarrow \quad \vec{x}' = \underbrace{\cancel{\Phi'\vec{u}}}_{= P(t)\Phi} + \Phi\vec{u}' \overset{\text{want}}{=} \cancel{P(t)\Phi\vec{u}} + \vec{f}$$

$$\Leftrightarrow \quad \vec{u}' = \Phi^{-1}\vec{f}$$

$$\Leftrightarrow \quad \vec{u}(t) = \int \Phi(t)^{-1}\vec{f}(t)\,dt + \vec{c}$$ (any antiderivative yields an $\vec{x}_p = \Phi\vec{u}$).

$$\Leftrightarrow \quad \vec{x}(t) = \Phi(t)\vec{u}(t)$$

$$\vec{x}(t) = \underbrace{\Phi(t)\vec{c}}_{\vec{x}_H} + \underbrace{\Phi(t)\int \Phi(t)^{-1}\vec{f}(t)\,dt}_{\vec{x}_p \text{, if you choose a particular antideriv.}}$$

Special case $P(t) = A$; const.
can be derived just as in Chapter 1 !!

$$\vec{x}' - A\vec{x} = \vec{f}$$

$$e^{-At}(\vec{x}' - A\vec{x}) = e^{-At}\vec{f}$$

$$\frac{d}{dt}\left(e^{-At}\vec{x}\right) = e^{-At}\vec{f}$$

$\int_0^t \% \, dt$:
$$e^{-At}\vec{x}(t) - \vec{x}_0 = \int_0^t e^{-As}\vec{f}(s)\,dt$$

$$\boxed{\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds}$$

[redacted]

Example 4 p. 363

$$\begin{cases} \vec{x}\,' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} -15 \\ -4 \end{bmatrix} t e^{-2t} \\ \vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \end{cases}$$

you could do this by hand, or...

Work for example 4 page 363

```
> A:=matrix(2,2,[4,2,3,-1]);
```

$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

```
> x0:=matrix(2,1,[7,3]);
```

$$x0 := \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

```
> f:=t->evalm(-t*exp(-2*t)*matrix(2,1,[15,4]));
  #the nonhomogeneous term.
```

$$f := t \to \mathrm{evalm}(-t\,\mathrm{e}^{(-2\,t)}\,\mathrm{matrix}(2, 1, [15, 4]))$$

```
> f(s);
```

$$\begin{bmatrix} -15\,s\,\mathrm{e}^{(-2\,s)} \\ -4\,s\,\mathrm{e}^{(-2\,s)} \end{bmatrix}$$

Maple (or I) currently has trouble directly evaluating the solution formula: I can't seem to make Maple integrate matrix valued functions without writing an explicit procedure (this did not used to be the case in older versions): The procedure below takes a matrix valued expression in s and returns a matrix expression in t, for which each entry is the integral of the input from s=0 to s=t:

```
> G:=proc(mat,m,n,s)
  local H, #matrix function to be returned
        i,  #row index
        j;  #column index

  H:=matrix(m,n):
   for i from 1 to m do
     for j from 1 to n do
     H[i,j]:=int(mat[i,j],s=0..t):
     od:
   od:
  return(evalm(H));
  end:
> integrand:=evalm(exponential(-A,s)&*f(s));
    #the integrand in the solution formula, page 4 notes or
    #page 363 text
```

$$integrand := \begin{bmatrix} -15\left(\frac{6}{7}\mathrm{e}^{(-5\,s)} + \frac{1}{7}\mathrm{e}^{(2\,s)}\right)s\,\mathrm{e}^{(-2\,s)} - 4\left(-\frac{2}{7}\mathrm{e}^{(2\,s)} + \frac{2}{7}\mathrm{e}^{(-5\,s)}\right)s\,\mathrm{e}^{(-2\,s)} \\ -15\left(-\frac{3}{7}\mathrm{e}^{(2\,s)} + \frac{3}{7}\mathrm{e}^{(-5\,s)}\right)s\,\mathrm{e}^{(-2\,s)} - 4\left(\frac{1}{7}\mathrm{e}^{(-5\,s)} + \frac{6}{7}\mathrm{e}^{(2\,s)}\right)s\,\mathrm{e}^{(-2\,s)} \end{bmatrix}$$

```
> sol:=t->evalm(exponential(A,t)&*(x0+G(integrand,2,1,s)));
    #solution formula
```

$$sol := t \to \mathrm{evalm}(\text{`\&*`}(\mathrm{exponential}(A, t), x0 + \mathrm{G}(integrand, 2, 1, s)))$$

```
> simplify(sol(t));   #see page 364!!!
```

$$\begin{bmatrix} \frac{3}{7}\mathrm{e}^{(-2\,t)} - \frac{1}{2}\mathrm{e}^{(-2\,t)}t^2 + \frac{46}{7}\mathrm{e}^{(5\,t)} + 2\,\mathrm{e}^{(-2\,t)}t \\ \frac{23}{7}\mathrm{e}^{(5\,t)} + \mathrm{e}^{(-2\,t)}t - \frac{2}{7}\mathrm{e}^{(-2\,t)} + \frac{3}{2}\mathrm{e}^{(-2\,t)}t^2 \end{bmatrix}$$

```
>
```