Math 2280-1
Tuesday 1/31

Office hours today postponed! (Sorry - I'm giving a 2270 guest lecture)
Instead, I will hold them from 1:40-2:30 today, in the computer lab.

Chapter 3.1-3.2 (HW from Chapter 3 will be due next Friday, Feb 10)
Here comes the linear algebra prerequisite!!

Linear differential equations of n^{th} order

Def: $L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$

is called a linear differential operator of order n

We assume the functions $a_j(x)$, $0 \le j \le n$ are defined and continuous for $x \in I$, some interval in $\mathbb{R}$.

Theorem 1 L is a linear transformation
from the vector space of n-times continuously differentiable functions $y(x)$ defined for $x \in I$ ("V")
with codomain equal to the continuous functions on I ("W")

proof: This amounts to showing

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad \forall y_1, y_2 \in V$$
$$L(cy_1) = cL(y_1) \qquad c \in \mathbb{R}$$

or more compactly,

$$L(c_1y_1 + c_2y_2) = c_1 L(y_1) + c_2 L(y_2)$$

proof:

examples: $L(y) = y' + p(x)y$ $\quad 1^{st}$ order
$L(y) = y'' + p(x)y' + q(x)y$ $\quad 2^{nd}$ order

Def Let the $a_j(x)$, I be as on page 1.
Then an n^{th} order linear differential eqtn is an DE of the form
$$L(y) = f(x)$$
where f is continuous on I.

e.g.
1^{st} order: $y' + p(x)y = q(x)$
2^{nd} order: $y'' + p(x)y' + q(x)y = f(x)$
etc.

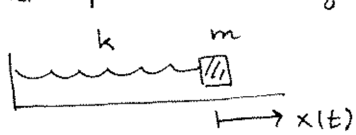
Theorem 2: Let $I \subset \mathbb{R}$ an interval
Let $x_0 \in I$
Let $a_0(x), a_1(x), \ldots a_{n-1}(x)$, $f(x)$ continuous on I
Then for any $b_0, b_1, \ldots b_{n-1} \in \mathbb{R}$
$\exists!$ solution to the initial value problem

$$\begin{cases} L(y) := y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = f(x) & \forall x \in I \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

(We'll discuss proof later in course)

Examples: $n=1$: $\begin{cases} y'(x) + p(x)y = q(x) \\ y(x_0) = y_0 \end{cases}$ We did this §1.5 !!

$n=2$ example: oscillating spring IVP



$$mx'' = \text{net forces} = \underbrace{-kx}_{\text{Hooke's const}} \underbrace{- cx'}_{\text{coeff of fric}} + \underbrace{f(t)}_{\text{external force}}$$

$$\begin{cases} x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{1}{m}f \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

if we specify initial
displacement and
velocity of spring configuration
we expect unique sol'n t for $t > 0$
(repeatable experiment!)

plausibility argument for Theorem 2:

Assume all functions are ∞'ly diff'ble and that $y(x)$ has a convergent

$\downarrow$

$a_j(x), f(x), y(x)$

Taylor series at x_0,

$$y(x) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{y^{(m)}(x_0)}{m!}(x-x_0)^m$$

From IVP we know coeffs at start of series

$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

from DE, $y^{(n)}(x_0) = f(x_0) - a_0(x_0)b_0 - a_1(x_0)b_1 - \cdots - a_{n-1}(x_0)b_n$

If we take $\frac{d}{dx}$ of DE, we can then solve for $y^{(n+1)}(x_0)$

then $\frac{d^2}{dx^2}$ of DE to get $y^{(n+2)}(x_0)$

recursively get all derivs of $y(x)$ at $x=x_0$, so the Taylor series for $y(x)$. !

Theorem 3 (Linear algebra)

Let $L : V \to W$ be a linear transformation.

Consider the problem of finding all solutions $\vec{v} \in V$ to the equation

$$L\vec{v} = \vec{b}$$

Let $\vec{v}_p$ be a (particular) solution to this equation.

Then every solution $\vec{v}$ can be written as

$$\vec{v} = \vec{v}_p + \vec{v}_H$$

where $\vec{v}_H$ is in the kernel of L, (i.e. solves the homogeneous equation $L\vec{v}_H = \vec{0}$)

(and each such $\vec{v} := \vec{v}_p + \vec{v}_H$ solves the original problem!)

proof: Let $\vec{v} = \vec{v}_p + \vec{v}_H$ where $L(\vec{v}_p) = \vec{b}$ and $L(\vec{v}_H) = \vec{0}$

Then
$$L(\vec{v}) = L(\vec{v}_p + \vec{v}_H) = L(\vec{v}_p) + L(\vec{v}_H) = \vec{b} + \vec{0} = \vec{b},$$
so $\vec{v}$ is a sol'n.

Now let $\vec{w}$ be any sol'n, i.e. $L(\vec{w}) = \vec{b}$

Then
$$\vec{w} = \vec{v}_p + \underbrace{(\vec{w} - \vec{v}_p)}_{= \vec{v}_H}$$
; note $L(\vec{w} - \vec{v}_p) = L(\vec{w}) - L(\vec{v}_p) = \vec{b} - \vec{b} = \vec{0}$

so $\vec{w}$ is the sum of $\vec{v}_p$ with a vector in the kernel of L

example 1: $L : \mathbb{R}^3 \to \mathbb{R}^2$
$$L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solve $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 1 & -2 & 3 & 1 \\ \hline 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = s$
$x_1 = 2s - 3t + 1$

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{“}\vec{x}_p\text{”}} + \underbrace{s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{“}\vec{x}_H\text{”}}$$

example 2: $L(y) := y' + 2y$

solve $y' + 2y = 3$ for $y(x)$, $x \in \mathbb{R}$

$$(e^{2x}y)' = e^{2x}3$$
$$e^{2x}y = \frac{3}{2}e^{2x} + C$$
$$y = \underbrace{\frac{3}{2}}_{\text{“}y_p\text{”}} + \underbrace{Ce^{-2x}}_{\text{“}y_H\text{”}}$$

For higher order DE's we will try to find the general solution by finding a y_P, and the general homogeneous sol'n

example 3 Explain why all sol'ns to

$$y'' - 4y = 1 \qquad (\text{for } x \in \mathbb{R})$$

are of the form

$$y(x) = \frac{1}{4} + c_1 e^{2x} + c_2 e^{-2x}$$

hint: after finding y_P, and at least part of y_H,
use $\exists!$ theorem to show every sol'n has claimed form.