Math 2280-1
Fri 27 Jan

HW for Fri Feb 3
(In addition to your Maple project)
§2.3 2, 3, 9, 11, 13, 14, 15, 16, 17, 18

projectiles revisited

vertical motion, height $y(t)$, velocity $v = \frac{dy}{dt}$

Newton:

$$m\frac{dv}{dt} = \underbrace{F_R}_{\text{air resistance}} + \underbrace{F_G}_{\text{gravity}}$$

linear model, $F_R = F_R(v) = -kv$

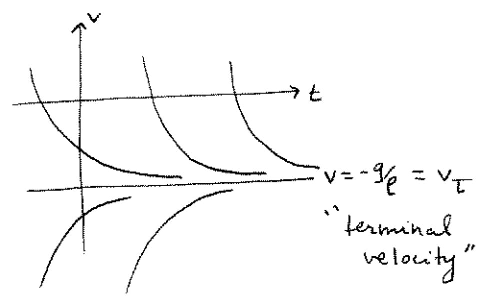
$F_G = -mg$

$$m\frac{dv}{dt} = -kv - mg$$

$$\frac{dv}{dt} = -\rho v - g \qquad \rho = k/m$$

: slope field for v: $\frac{dv}{dt} = -\rho\left(v + \frac{g}{\rho}\right)$

equil sol'n $v = -g/\rho$



solve by linear DE

solve by separable DE

$$\frac{dv}{dt} + \rho v = -g$$

$$\left(e^{\rho t} v\right)' = -g e^{\rho t}$$

$$e^{\rho t} v = -\frac{g}{\rho} e^{\rho t} + C$$

$$v_0 = -\frac{g}{\rho} + C; \quad C = v_0 + g/\rho$$

$$e^{\rho t} v = -\frac{g}{\rho} e^{\rho t} + v_0 + g/\rho$$

$$\boxed{\begin{aligned} v &= -\frac{g}{\rho} + \left(v_0 + g/\rho\right) e^{-\rho t} \\ v &= v_\tau + (v_0 - v_\tau) e^{-\rho t} \end{aligned}}$$

$v_\tau := -g/\rho$ terminal velocity

$$y(t) = \int v(t)\,dt = t v_\tau + \frac{1}{\rho}(v_\tau - v_0) e^{-\rho t} + C$$

$$y_0 = \frac{1}{\rho}(v_\tau - v_0) + C; \quad C = y_0 - \frac{1}{\rho}(v_\tau - v_0)$$

$$y(t) = t v_\tau + \frac{1}{\rho}(v_\tau - v_0) e^{-\rho t} + y_0 - \frac{1}{\rho}(v_\tau - v_0)$$

; collect terms : $\boxed{y(t) = y_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)\left(1 - e^{-\rho t}\right)}$

Examples 1 & 2 p. 98–101

crossbow bolt

Example 2

$y_0 = 0$

$v_0 = 49$ m/s

$g = 9.8$ m/s^2

$v_T = -245$ m/s (measured)

$v_T = -g/\rho \Rightarrow \rho = \frac{g}{-v_T} = \frac{9.8}{245} = \boxed{.04 = \rho}$

$\frac{dv}{dt} = -\rho v - g$

from page 1

$v = -245 + (49+245)e^{-.04t}$

$v = -245 + 294\, e^{-.04t}$

$y = -245t + 25(294)(1 - e^{-.04t})$

max ht when $v = 0$;

$e^{-.04t} = \frac{245}{294}$

$t = \frac{\ln(294/245)}{.04} \approx \boxed{4.55 \text{ sec}}$, $y_{max} = y(4.55) \approx \boxed{108 \text{ m}}$

bolt lands when $y(t) = 0$

> solve $(y(t)=0, t)$;

$t = 0, \boxed{9.4}$; $\boxed{v(9.4) \approx -43.2 \text{ m/s}}$

$y(0) = 0$

Example 1 no drag; $\frac{dv}{dt} = -g$

$y_0 = 0$

$v_0 = 49$ m/s

$g = 9.8$ m/s^2

$v = -gt + v_0$

$v = -9.8t + 49$

$y = -4.9t^2 + 49t \quad (+\overset{0}{\overset{\shortparallel}{y_0}})$

max ht when $v = 0$;

$\boxed{t = \frac{49}{9.8} = 5 \text{ sec}}$

$y(5) = 49[(.1)(25)(-1) + 5]$

$= \boxed{(2.5)(49) = 123.5 \text{ m}}$

lands after 10 secs, with velocity $\boxed{-49 \text{ m/s}}$

Math 2280-1
Friday January 27
Computation sheet
Example 2, section 2.3

```
> y:= t-> -245*t + 294*25*(1 - exp(-.04*t));
     #height y(t)
```

$$y := t \to -245\,t + 7350 - 7350\,e^{(-0.04\,t)}$$

```
> v:= t -> 294*exp(-.04*t) - 245;
     #velocity v(t)
```

$$v := t \to 294\,e^{(-0.04\,t)} - 245$$

```
> 25*ln(294.0/245);
     #by hand we can set v(t)=0 and solve for t:
```

4.558038920

```
> solve(v(t)=0,t);
     #or we can ask Maple to do it:
```

4.558038920

```
> y(4.558038920);
     #max height
```

108.280465

```
> solve(y(t)=0,t);
     #find when returns to ground
```

9.410949931, 0.

```
> 9.410949931 - 4.558038920;
     #time descending
```

4.852911011

```
> v(9.410949931);
     #speed when it lands
```

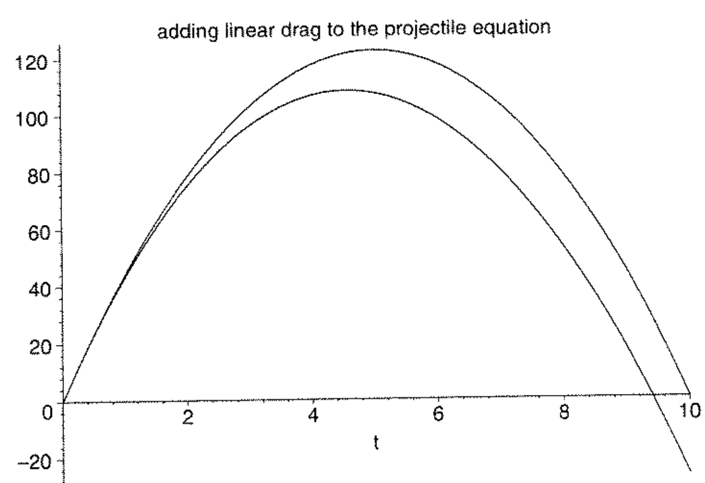
-43.2273093

Conclusions: bolt rises for 4.56 seconds, to a height of 108.3 meters. Then it spends 4.85 seconds descending, landing with a velocity of -43.3 meters per second. We can see these facts graphically if we plot:

```
> with(plots):
Warning, the name changecoords has been redefined
> z:= t->-4.9*t^2 + 49*t;
     #the no drag solution
```

$$z := t \to -4.9\,t^2 + 49\,t$$

```
> plot({z(t),y(t)}, t = 0..10, color=black,
  title=`adding linear drag to the projectile equation`);
```



other drag models (non-linear).

text says that experimentally (depending on object and range of velocities),

$$|F_R| = k|v|^p \qquad 1 \le p \le 2$$

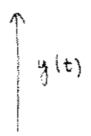
we just did $p=1$.

another case which may be solved explicitly (see HW) is $p=2$.

$$m\frac{dv}{dt} = F_G + F_R = -mg \underbrace{- kv|v|}$$

so F_R is in opposite dir. of velocity.

going up: $(v>0)$



$$\frac{dv}{dt} = -g - \rho v^2 = -g\left(1 + \tfrac{\rho}{g}v^2\right) \qquad \left(\rho = \tfrac{k}{m} \text{ still}\right)$$

$$\frac{dv}{1+\frac{\rho}{g}v^2} = -g\,dt$$

(eqtn 13 p.101)

use arctan!

going down: $(v<0)$

$$\frac{dv}{dt} = -g + \rho v^2 = -g\left(1 - \tfrac{\rho}{g}v^2\right)$$

$$\frac{dv}{1-\frac{\rho}{g}v^2} = -g\,dt$$

(eqtn 16 p.102)

use arctanh, int tables, or partial fractions!

In the crossbow example, text discusses how the $p=1$, $p=2$ predictions are almost identical, (if ρ is modified appropriately).