Math 2280

Monday 1/23 §2.1 & 2.2

Here are the details we filled in on Friday, when we solved the logistic eqtn DE

$$\begin{cases} \dfrac{dP}{dt} = kP(M-P) \\ P(0)=P_0 \end{cases}$$

$$\frac{dP}{P(P-M)} = -k\,dt$$

$$\frac{1}{P(P-M)} = -\frac{1}{M}\left(\frac{1}{P}-\frac{1}{P-M}\right)$$

(guess and correct partial fractions ... you can also do it the long way:

$$\frac{1}{P(P-M)} = \frac{A}{P}+\frac{B}{P-M} = \frac{A(P-M)+BP}{P(P-M)}$$

so $1 = A(P-M)+BP$

@ $P=M \Rightarrow 1 = MB \Rightarrow B = \frac{1}{M}$

@ $P=0 \Rightarrow 1 = -AM \Rightarrow A = -\frac{1}{M}$ ✓

& take minus:

$$\int \frac{1}{M}\left(\frac{1}{P}-\frac{1}{P-M}\right)dP = \int k\,dt$$

$$\frac{1}{M}\Big[\ln|P| - \ln|P-M|\Big] = kt + C$$

$$\ln\left|\frac{P}{P-M}\right| = Mkt + \tilde{C}$$

exponentiate:

$$\left|\frac{P}{P-M}\right| = \tilde{\tilde{C}}\, e^{Mkt}$$

$P(0)=P_0 \Rightarrow \tilde{\tilde{C}} = \left|\frac{P_0}{P_0-M}\right|$, then reason the removable of abs val signs (would get ±, but must be + because of IVP)

$$\Rightarrow \frac{P}{P-M} = \frac{P_0}{P_0-M}e^{Mkt}$$

$$P = (P-M)\left[\frac{P_0}{P_0-M}e^{Mkt}\right]$$ (multiply thru)

$$P\left(1-\frac{P_0}{P_0-M}e^{Mkt}\right) = \frac{-MP_0}{P_0-M}e^{Mkt}$$

$$P(t) = \frac{\frac{-MP_0}{P_0-M}e^{Mkt}}{1-\frac{P_0}{P_0-M}e^{Mkt}} \cdot \frac{e^{-Mkt}(P_0-M)}{e^{-Mkt}(P_0-M)}$$

↓ just making formula prettier from here on down

$$= \frac{-MP_0}{(P_0-M)e^{-Mkt}-P_0} \cdot \frac{\frac{1}{P_0}(-1)}{\frac{1}{P_0}(-1)}$$

$$\boxed{P(t) = \frac{M}{\frac{M-P_0}{P_0}e^{-Mkt}+1} = \frac{MP_0}{(M-P_0)e^{-Mkt}+P_0}}$$

Friday

example : Maple handout explains how logistic eqtn has been used to model U.S. populations - we should discuss this.

§2.2 : A general 1st order DE $\frac{dx}{dt} = f(t,x)$

is called autonomous if $\frac{dx}{dt} = f(x)$ ($\frac{dx}{dt}$ only depends on x itself, not also on t)

def x(t) is an equilibrium sol'n to a DE iff $x(t) \equiv C$, a constant

If the DE is autonomous and $x(t) \equiv C$ is an equilibrium soltn, then

$$0 = \frac{dx}{dt} = f(x) = f(C) = 0.$$

And if $f(C)=0$ then $x(t) \equiv C$ is an equilibrium sol'n.

$$\boxed{\text{equilibrium sol's of } \frac{dx}{dt} = f(c) \text{ are exactly the fcns } x(t) \equiv c \text{ where } f(c)=0}$$

example

$$\frac{dx}{dt} = kx(M-x)$$

$x \equiv 0$, $x \equiv M$ are the equil. sol'ns

example find the equil. sol'ns of

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

Def Let c be an equilibrium sol'n for a deqtn. Then

c is stable iff $\forall \varepsilon > 0 \ \exists \ \delta > 0$ s.t. for sol'ns $x(t)$ with $|x(0) - c| < \delta$ we have $|x(t) - c| < \varepsilon \quad \forall t > 0$

(sol'ns which start close enough to c stay arbitrarily close to it.)

c is unstable if it is not stable

example: Make the slope field (or alternately, sketch a sufficient sample of sol'n graphs) and phase portrait for

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

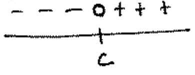
and discuss stability of the equil sol'ns $x = 0, x = -1$

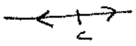
example find equilibria and discuss stability for $\frac{dx}{dt} = 3x - x^2$

Def c is called asymptotically stable (stronger requirement than stable)
iff $\exists \ \delta > 0$ s.t. $|x(t) - c| < \delta \Rightarrow \lim_{t \to \infty} x(t) = c$.

Are any of the equilibria you found above asymptotically stable?

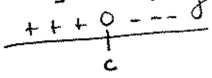
Theorem: Consider $\frac{dx}{dt} = f(x)$, f continuously differentiable.

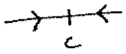
if $f(c) = 0$ then $f'(c) > 0 \Rightarrow$


sign f $\Rightarrow$


Unstable

f increasing

$f'(c) < 0 \Rightarrow$


sign f $\Rightarrow$


stable

f decreasing

$f'(c) = 0$ You must do additional work.

More "real" examples

doomsday extinction:

$$\frac{dP}{dt} = aP^2 - bP \qquad a, b > 0.$$

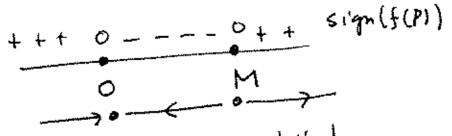
$$= kP(P-M) \qquad \begin{matrix} k = a \\ -kM = b \end{matrix}$$

e.g. if $\beta(t)$ (fertility rate) $\overset{\text{proportional to}}{\sim} P$

(e.g. herd roams in a fixed area and mates randomly upon meeting opposite-sex mate)

then $B(t) = aP^2$;

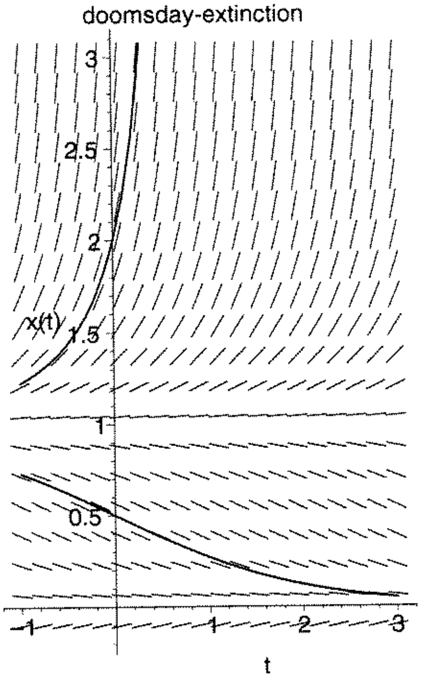
perhaps $D(t) = -bP$.



0 stable; M Unstable!

If $0 < P_0 < M$ then $\lim_{t \to \infty} P(t) = 0$ (so in real life, extinction)

If $P_0 > M$ then $\exists\ t_1 < \infty$ s.t. $\lim_{t \to t_1} P(t) = +\infty$ (doomsday!)



example

$$\begin{cases} \dfrac{dx}{dt} = x(x-1) \\ x(0) = 2 \end{cases}$$

Find the time of doomsday.

(ans $t = \ln 2$!)