Monday 1/23

Here are the details we filled in on Friday, when we solved the logistic eqtn DE

\[
\begin{align*}
\frac{dP}{dt} &= kP(M-P) \\
P(0) &= P_0
\end{align*}
\]

\[
\frac{dP}{P(P-M)} = -kd\text{dt}
\]

\[
\int \frac{1}{M(P-M)}dP = \int \frac{1}{P} \frac{dP}{P-M}
\]

\[
\ln |P| - \ln |P-M| = kt + C
\]

\[
\ln |\frac{P}{P-M}| = Mkt + C
\]

Exponentiate:

\[
\left| \frac{P}{P-M} \right| = e^{Mkt + C}
\]

\[
P(0) = P_0 \Rightarrow e^{C} = \left| \frac{P_0}{P_0-M} \right|
\]

\[
\Rightarrow \frac{P}{P-M} = \frac{P_0}{P_0-M} e^{Mkt}
\]

\[
P = (P-M) \left[ \frac{P_0}{P_0-M} e^{Mkt} \right]
\]

\[
P(t) = M\frac{e^{Mkt}}{P_0-M} \left[ \frac{P_0 e^{Mkt}}{P_0-M} - P_0 \right] - \frac{P_0}{P_0-M} e^{Mkt}
\]

\[
P(t) = \frac{M}{P_0 e^{Mkt} + 1} = \frac{MP_0 e^{Mkt} + P_0}{(M-P_0) e^{Mkt} + P_0}
\]
62.2: A general 1st order DE \( \frac{dx}{dt} = f(t, x) \)

is called autonomous if \( \frac{dx}{dt} = f(x) \) (\( \frac{dx}{dt} \) only depends on \( x \) itself, not also on \( t \)).

Def \( x(t) \) is an equilibrium sol'n to a DE iff \( x(t) = C \), a constant.

If the DE is autonomous and \( x(t) = C \) is an equilibrium sol'n, then

\[
0 = \frac{dx}{dt} = f(x) = f(C) = 0.
\]

And if \( f(C) = 0 \) then \( x(t) = C \) is an equilibrium sol'n.

**equilibrium sol's of**

\[
\frac{dx}{dt} = f(x)
\]

are exactly the

\[
\text{funcs } x(t) \equiv c
\]

where \( f(c) = 0 \).

**example**

\[
\frac{dx}{dt} = kx(M-x)
\]

\[
\begin{align*}
&x=0 \\
&x=M
\end{align*}
\]

are the equil. sol's

**example** find the equil. sol's of

\[
\frac{dx}{dt} = x^3 + 2x^2 + x
\]
Def. Let $c$ be an equilibrium solution for a depth. Then

$c$ is **stable** iff $\exists \delta > 0$ s.t. for solutions $x(t)$ with $|x(\theta) - c| < \delta$
we have $|x(t) - c| < \delta$ for all $t > \theta$.

(solutions which start close enough to $c$ stay arbitrarily close to it.)

$c$ is **unstable** if it is not stable.

**Example:** Make the slope field (or alternatively, sketch a sufficient sample of solution graphs) and phase portrait for

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

and discuss stability of the equilibria $x = 0, x = -1$.

**Example:** Find equilibria and discuss stability for

$$\frac{dx}{dt} = 3x - x^2$$

Def. $c$ is called **asymptotically stable** (stronger requirement than stable)

iff $\exists \delta > 0$ s.t. $|x(t) - c| < \delta \Rightarrow \lim_{t \to \infty} x(t) = c.$

Are any of the equilibria you found above asymptotically stable?

**Theorem:** Consider $\frac{dx}{dt} = f(x)$, $f$ continuously differentiable.

If $f(c) = 0$ then

- $f'(c) > 0 \Rightarrow$ increasing
- $f'(c) < 0 \Rightarrow$ decreasing
- $f'(c) = 0 \Rightarrow$ You must do additional work.

$\rightarrow$ **stable**

$\rightarrow$ **unstable**
More "real" examples

Doomsday extinction:

\[
\frac{dP}{dt} = aP^2 - bP + kP(P-M) \quad a,b>0, \quad k=a-bM/b
\]

$\downarrow$

Example

\[
\begin{cases}
\frac{dx}{dt} = x(x-1) \\
x(0) = 2
\end{cases}
\]

Find the time of doomsday.

(Ans: $t = \ln 2$)

\[
\begin{array}{c}
\text{doomsday-extinction}
\end{array}
\]

If $0 < P_0 < M$ then $\lim_{t \to \infty} P(t) = 0$ (so in real life, extinction)

If $P_0 > M$ then $\exists t, < \infty$ s.t. $\lim_{t \to \infty} P(t) = +\infty$ (doomsday!)

\[b(t) \quad \text{(fertility rate)} \quad \sim P\]

(e.g. herd roams in a fixed area and mates randomly upon meeting opposite-sex mate)

Then $B(t) = aP^2$;

perhaps $D(t) = bP$. 

O stable; M Unstable!