Name..
I.D. number...

Math 2280-1
FINAL EXAM
May 1, 2006

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Use Laplace Transform techniques to solve the initial value problem for a resonating spring:

$$x''(t) + 4\,x(t) = 3\sin(2\,t)$$
$$x(0) = x_0$$
$$x'(0) = v_0\,.$$

(15 points)

2) You will be using the following matrix A in problems (3) and (4). In particular you will need the eigenvalues and eigenvectors. Here is the matrix:

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

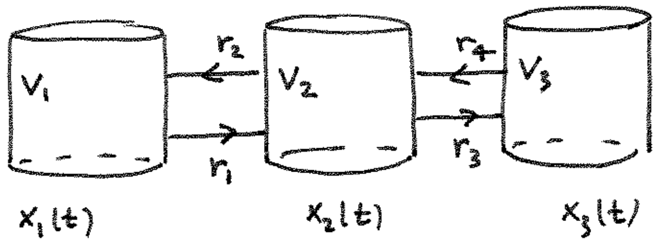
2a) Find the characteristic polynomial of A, and factor it to show that the eigenvalues of A are 0, -3, -1.

(5 points)

2b) A basis for the $\lambda = 0$ eigenspace is the vector [1,1,1]. Find bases for the other two eigenspaces, and exhibit all three vectors together as an eigenbasis for A. Make sure to check your answers carefully, as these eigenvalues and eigenvectors will be used later.

(10 points)

3) Consider the following three-tank configuration. Let tank i have volume $V_i(t)$ and solute mass $x_i(t)$, at time t. Well-mixed liquid flows between tanks 1 and 2, and also between tanks 2 and 3, with rates r_1, r_2, r_3, r_4, as indicated in the diagram.



3a) What are the 6 differential equations governing the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$, and the solute masses $x_1(t)$, $x_2(t)$, $x_3(t)$?

(4 points)

3b) Suppose all four rates are 100 gallons/hour, so that the volumes remain constant. Suppose all three volumes are 100 gallons. Show that in this case the three differential equations in 3a) for the solute masses yield the system

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(4 points)

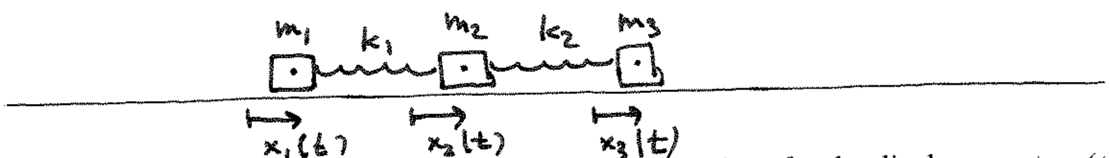
3c) Continuing with the system from (3b), if tank 1 initially had 10 kg of solute, tank 2 had 20 kg, and tank 3 had no solute, what do you expect the limiting solute amounts per tank to be, as time approaches infinity? Explain your reasoning.

(2 points)

3d) Solve the initial value problem for (3b), using the initial values in (3c). Make use of the fact that the matrix in this system is exactly the one you found eigenvectors and eigenvalues for already, in problem 2. (Don't repeat the work from that problem, use the results!)

(10 points)

4) Consider the following configuration of three masses held together with two springs, with positive displacements from equilibrium measured to the right, as usual. Notice that this train of springs is not anchored to any wall!



4a) Derive the system of three second order differential equations for the displacements $x_1(t)$, $x_2(t)$, $x_3(t)$ using Newton's and Hooke's Laws.

(6 points)

4b) Show that in case all three masses m are identical and both springs constants k are also equal, and if mass and force units are chosen so that m=k, then the system in part (4a) becomes

$$\begin{bmatrix} \frac{d^2}{dt^2}x_1(t) \\ \frac{d^2}{dt^2}x_2(t) \\ \frac{d^2}{dt^2}x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(2 points)

4c) Let B be any n by n matrix which has an eigenvector **v** with eigenvalue zero. Show that

$$y(t) := v$$
$$z(t) := t\,v$$

both solve the second order homogeneous system

$$x''(t) = B\,x$$

(5 points)

4d) What is the dimension of the solution space for the spring system in (4b)? Explain!

(3 points)

4e) Write down the general solution to this spring system from (4b), making use of the fact that the matrix A from problem (2) has appeared again!

(6 points)

4f) Describe the motions associated to each of the fundamental modes in this spring system.

(3 points)

5) Consider a population x(t) governed by the differential equation

$$x'(t) = 7x - x^2 .$$

5a) Find the equilibrium solutions and draw the phase diagram for this population model. Indicate which equilibrium solutions are stable and which are unstable.

(5 points)

5b) Is this a logistic or doomsday-extinction model? Explain.

(2 points)

5c) Solve the initial value problem

$$x'(t) = 7x - x^2$$
$$x(0) = 14$$

(8 points)

6) Consider the system of differential equations below which models two populations x(t) and y(t). (You can think of this as an extension of problem (5) for the population x(t), which now finds itself in the presence of another species y(t).)

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 7x - x^2 - xy \\ -5y + xy \end{bmatrix}$$

6a) If this was a model of two interacting populations, which model would it be? Explain.

(2 points)

6b) Find the equilbrium solutions to this system of differential equations.

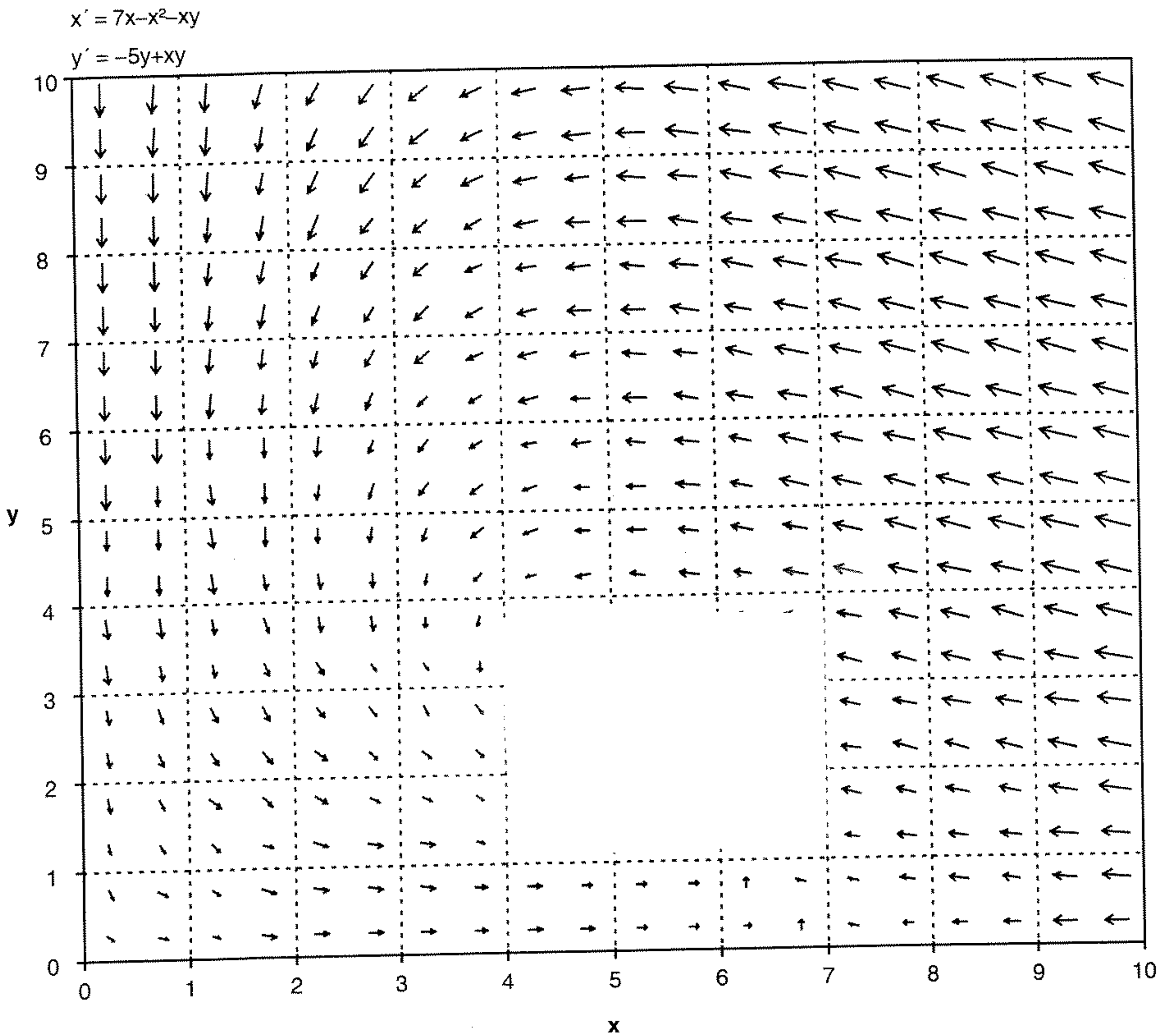
(4 points)

6c) Only one of your equilbrium solutions has postive populations of both species. Linearize the population model near this equilibrium point. Use your analysis to classify which type of equilbrium point this is.

(8 points)

6d) Using your work from (6c), fill in the missing piece of the pplane phase portrait below. Also draw the the phase diagrams along each of the positive x and y axes. Then make a prediction about long term behavior of solutions to this population model, assuming both populations start out positive. (Depending on your analysis in (6c) your prediction may or may not depend on where in the first quadrant the initial population vector is located.)

(6 points)



7) Let f(t) be a 2L-periodic function. Recall that the Fourier series for f is given by

$$f \sim \frac{1}{2} a_0 + \left(\sum_{n=1}^{\infty} a_n \cos\left(\frac{n \pi t}{L}\right) \right) + \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n \pi t}{L}\right) \right)$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n \pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n \pi t}{L}\right) dt$$

7a) Let f(t) be the sawtooth function with period 2π, defined by $f(t) = t$ for $-\pi < t < \pi$. Sketch the graph of f.

(3 points)

7b) Prove that f(t) has Fourier series

$$f \sim 2 \left(\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} \sin(n\, t)}{n} \right)$$

(7 points)

Hint:

$$\int t \sin(a\, t)\, dt = \frac{\sin(a\, t) - a\, t \cos(a\, t)}{a^2} + C$$

7c) Consider the same spring configuration as in problem (1), but now force with the sawtooth function f(t).

$$x''(t) + 4\,x((t) = f(t)$$

Explain why this differential equation exhibits resonance, even though the period of f(t) is not the natural period of the unforced system.

(5 points)

7d) Assume infinite superposition is valid in order to write down the general solution to the forced oscillator in (7c). Hints: don't forget the homogeneous solution, and for the resonating piece of your parcticular solution you can quote any work you did problem (1).

(10 points)

8) Consider the wave equation

$$y_{tt} = 25\, y_{xx}$$

for a function y(x,t), with interval $0 < x < \pi$, and $t > 0$.
Solve this wave equation for $t > 0$, with fixed endpoint boundary conditions,

$$y(0, t) = 0 = y(\pi, t)$$

with initial displacement

$$y(x, 0) = 2 \sin(3\, x)$$

and initial velocity

$$y_t(x, 0) = -3 \sin(2\, x),$$

for $0 < x < \pi$.

(10 points)

9) The existence and uniqueness theorem for linear first order systems of differential equations says that if A(t) is a continuous, n by n matrix-valued function of t, and f(t) is a continuous n-vector valued function, each on the in t-interval I, and if t_0 is a point in I, then each initial value problem

$$x'(t) = A(t)\,x(t) + f(t)$$
$$x(t_0) = x_0$$

has a unique solution x(t), defined for all t in I. Use this theorem to explain (i.e. prove) why the solution space to the homogeneous system

$$x'(t) = A(t)\,x(t)$$

is n-dimensional.

(10 points)

10a) What is Euler's formula?

(2 points)

10b) Show that the addition angle formulas for sine and cosine are equivalent to the identity

$$\mathbf{e}^{(i(\alpha+\beta))} = \mathbf{e}^{(i\alpha)}\,\mathbf{e}^{(i\beta)}$$

(3 points)