Math 2280-1
Wed 2/8

Need to postpone experiments :(

- Do simple harmonic motion example, pages 4-5 Tues-notes (p. 186 text)

- now, add damping:

$$mx'' + cx' + kx = 0$$

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$$

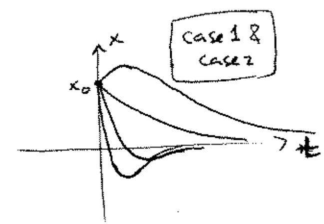
$$x'' + 2px' + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}} \text{ (as before), = undamped angular frequency (rad/time)}$$

$$p = \frac{c}{2m}$$

e^{rt}: $r^2 + 2pr + \omega_0^2 = 0$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$



Case 1 $p^2 > \omega_0^2$: $-p - \sqrt{p^2 - \omega_0^2} < -p + \sqrt{p^2 - \omega_0^2} < 0$

$$r_2 < r_1 < 0$$

OVERDAMPED

$$x_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$= e^{r_1 t}\left[c_1 + c_2 e^{(r_2 - r_1)t}\right]$$

at most one root & decays exponentially in time (Figure 3.4.7 p 188)

Case 2 $p^2 = \omega_0^2$: double root $r = -p$

CRITICALLY DAMPED

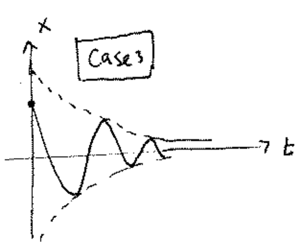
$$x_H(t) = c_1 e^{-pt} + c_2 t e^{-pt}$$
$$= e^{-pt}(c_1 + c_2 t)$$ also ; Figure 3.4.8.

Case 3 $p^2 < \omega_0^2$, $r = -p \pm i\underbrace{\sqrt{\omega_0^2 - p^2}}_{\omega_1 := \sqrt{\omega_0^2 - p^2}}$

$$x_H(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$$
$$= e^{-pt}(C\cos(\omega_1 t - \alpha))$$

UNDERDAMPED

decays exponentially but oscillates ∞'ly often.



Ce^{-pt} is called "time-varying amplitude"
ω_1 is called "pseudo angular frequency"
etc.

notice $\omega_1 < \omega_0$, i.e. damping retards the oscillation.

Example 2 p. 189 contrasted to Example 1:

$$\begin{cases} x'' + 2x' + 100x = 0 \\ x_0 = 1 \quad m \\ v_0 = -5 \quad m/s \end{cases} \qquad \begin{cases} x'' + 100x = 0. \\ x_0 = 1 \quad m \\ v_0 = -5 \quad m/s \end{cases}$$

sol'n: e^{rt}: $r^2 + 2r + 100 = 0$

$$(r+1)^2 + 99 = 0$$

$$(r + 1 + \sqrt{99}\,i)(r + 1 - \sqrt{99}\,i) = 0$$

$$r = -1 \pm i\sqrt{99}$$

$$x_H(t) = e^{-t}\left(A\cos\sqrt{99}\,t + B\sin\sqrt{99}\,t\right) \qquad \omega_1 = \sqrt{99} < \omega_0 = 10$$

$x_0 = 1 \Rightarrow A = 1$

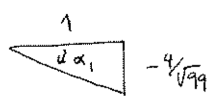
$v_0 = -5 \Rightarrow -A + \omega_1 B = -5$

$$\omega_1 B = -4$$

$$B = -\frac{4}{\omega_1} \qquad \left(\text{undamped } B = -\frac{5}{\omega_0}\right)$$

$$= \frac{-4}{\sqrt{99}}$$

$$C = \sqrt{A^2 + B^2} = \sqrt{1 + \frac{16}{99}} \approx 1.078$$



$$\alpha_1 = \arctan\left(\frac{-4}{\sqrt{99}}\right) \approx -.382$$

see picture!

Math 2270-1
Damping added to Example 1, i.e.
Example 2 page 189

```
> x1:=t->exp(-t)*(cos(sqrt(99)*t)-4/sqrt(99)*sin(sqrt(99)*t));
```

$$x1 := t \to \mathbf{e}^{(-t)}\left(\cos(\sqrt{99}\,t) - \frac{4\sin(\sqrt{99}\,t)}{\sqrt{99}}\right)$$

```
> C1:=sqrt(1+16/99.);  #amplitude
  alpha1:=arctan(-4/sqrt(99.));  #phase
  alpha1+evalf(2*Pi); #book's phase
  omega1:=sqrt(99.):  #pseudo angular frequency
  nu1:=evalf(omega1/(2*Pi));  #pseudofrequency
  T1:=1/nu1;  #pseudoperiod
  delta1:=alpha1/omega1; #pseudolag
```

$$C1 := 1.077782985$$

$$\alpha 1 := -0.3822423467$$

$$5.900942961$$

$$\nu 1 := 1.583571689$$

$$T1 := 0.6314838835$$

$$\delta 1 := -0.03841680130$$

```
> y1:=t->C1*exp(-t)*cos(omega1*t-alpha1); #same function!

>
```

$$y1 := t \to C1\,\mathbf{e}^{(-t)}\cos(\omega 1\, t - \alpha 1)$$

```
> with(plots):
  plot({x1(t),y1(t),x(t),C1*exp(-t),-C1*exp(-t)},t=0..4,color=black,
  title=`simple harmonic and underdamped
  motion`);
```