Math 2280-1
Tues 2/6

Hey! wanna try some HW?

For your reference, a mixture of old and new facts about complex exponentials.

Old

(1) Euler's formula. (2270)

$$e^{i\theta} := \cos\theta + i\sin\theta$$

reason: $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ converges for all real x.
[you can show it also converges for all $z = x+iy$].

so, formally at least,

$$e^{i\theta} = \underbrace{1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots}_{\text{real part is Taylor series for } \cos\theta} + \frac{(i\theta)^n}{n!} + \cdots$$

imaginary part adds up to $i\sin\theta$

(2) In 2270 we checked

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$, using addition angle formulas

(And since you expect this formula to be true, it gives a way to recall addition angle formulas)

$$e^{i(\alpha+\beta)} := \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$e^{i\alpha}e^{i\beta} := (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \Big[\underbrace{\cos\alpha\cos\beta - \sin\alpha\sin\beta}_{\cos(\alpha+\beta)}\Big] + i\Big[\underbrace{\cos\alpha\sin\beta + \sin\alpha\cos\beta}_{\sin(\alpha+\beta)}\Big]$$

New: Def $\frac{d}{dx}\left[f(x) + i g(x)\right] := f'(x) + i g'(x)$

Def $e^{\alpha+\beta i} := e^{\alpha} e^{\beta i} = e^{\alpha}(\cos\beta + i\sin\beta)$

Thm if $r = a + bi$ then

$$\frac{d}{dx} e^{rx} = r e^{rx}$$

Check:

This leads to theorem on page 3 Monday, which we already discussed.
(and did an example)

Recap : let $L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y \qquad a_j \text{ const.}$

$p(r) = r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$

if r_j is a real root of p, $p(r_j) = 0 \implies y = e^{r_j x}$ solves $L(y) = 0$

if $(r - r_j)^{k_j}$ is a factor of $p \implies$

$$\left.\begin{aligned} y_1 &= e^{r_j x} \\ y_2 &= x e^{r_j x} \\ &\vdots \\ y_{k_j} &= x^{k_j - 1} e^{r_j x} \end{aligned}\right\} \text{ are } k_j \text{ lin ind soltns to } L(y) = 0$$

if $r_j = a + bi$ is a complex root, $p(a+bi) = 0 \implies$

$$\left.\begin{aligned} y_1 &= e^{ax} \cos bx \\ y_2 &= e^{ax} \sin bx \end{aligned}\right\} \text{ are l.i. soltns to } L(y) = 0$$

if $(r - r_j)^{k_j}$ is a factor of p

also get $\quad x e^{ax} \cos bx, \quad x e^{ax} \sin bx$

$$\vdots \qquad\qquad \vdots$$

$$x^{k_j - 1} e^{ax} \cos bx, \quad x^{k_j - 1} e^{ax} \sin bx$$

In this way we can always find a basis for $\ker(L)$,
(assuming we can figure out how to factor p!)

§3.4 : Applications of const. coeff homogeneous DE's.

The spring, the pendulum.

$x(t)$, displacement of mass from equilibrium

spring, mass, dashpot configuration.

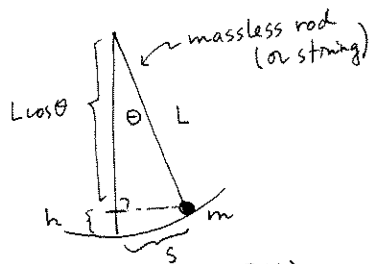
k = "Hooke's constant"
c = "coefficient of friction

Newton + linearization:

$$m x'' = -kx - cx'$$

$$\boxed{m x'' + c x' + k x = 0}$$

but very similar mathematically is the <u>pendulum equation</u>



$\theta(t)$

mass moves in a circular arc

$s(t) = L\theta(t)$ is (signed) distance to "bottom

$v(t) = s'(t) = L\theta'(t)$

neglecting drag this a conservative system

so total energy = KE + PE = constant

$$= \tfrac{1}{2} m \left(L\theta'(t)\right)^2 + mgh$$

$$= \tfrac{1}{2} m L^2 \theta'(t)^2 + mg L(1-\cos\theta)$$

$$\frac{d}{dt}(KE+PE) \equiv 0$$

$$\Rightarrow mL\left(\theta'\theta'' L + g\sin\theta\,\theta'\right) \equiv 0$$

$$mL\theta'\left(\underbrace{L\theta'' + g\sin\theta}_{\text{so } = 0.}\right) \equiv 0$$

$mL\theta'$: only zero at isolated instances

Linearize! (near $\theta = 0$)
$\sin\theta \approx \theta$

$$\boxed{\begin{aligned} L\theta'' + g\theta &= 0 \\ \theta'' + \tfrac{g}{L}\theta &= 0 \end{aligned}}$$

(We could add drag and get
$L\theta'' + c\theta' + g\theta = 0$,
"same" DE as for mass-spring!)

<u>Case 1</u> free undamped motion

$$m x'' + kx = 0 \qquad (\text{or } L\theta'' + g\theta = 0)$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0 \qquad \left(\omega_0 := \sqrt{\tfrac{k}{m}}\right)$$

for $x(t) = e^{rt}$,

$$L(x) = (r^2 + \omega_0^2)e^{rt} = 0 \text{ iff } r = \pm i\omega_0$$

$$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$$

since trig fcns eat radians, if t is measured in sec, ω_0's units are radians/sec

ω_0 is called angular frequency

so

$$x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$= C\cos(\omega_0 t - \alpha)$$

(using trig identities)

$x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \alpha)$

$= C\cos(\omega_0(t-\delta))$

simple harmonic motion

$C :=$ amplitude

$\alpha :=$ phase ; $\delta = \alpha/\omega_0 =$ "lag"

$\omega_0 :=$ angular frequency (rad/sec)

$\nu = f :=$ frequency (cycles/sec or herz)

$= \frac{\omega_0}{2\pi}$

$T :=$ period (secs/cycle) $= \frac{1}{f} = \frac{2\pi}{\omega_0}$

$C\cos(\omega_0 t - \alpha)$

$= C\cos\omega_0 t\cos\alpha + C\sin\omega_0 t\sin\alpha$

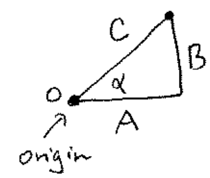
$= A\cos\omega_0 t \quad + B\sin\omega_0 t$

Thus

$C\cos\alpha = A$
$C\sin\alpha = B$ $\rightsquigarrow$ so $C^2 = A^2 + B^2$

$\cos\alpha = A/C$
$\sin\alpha = B/C$

summarized by



depending on the signs of A&B the hypotenuse can point into any quadrant!

if $-\pi/2 < \alpha < \pi/2$,
$\alpha = \arctan(B/A)$

if $0 < \alpha < \pi$
$\alpha = \arccos(A/C)$

if $-\pi/2 < \alpha < \pi/2$
$\alpha = \arcsin(B/C)$

in third quad ($\pi < \alpha < 3\pi/2$),
$\alpha = \pi + \arctan B/A$ works.

Example 1 page 186

$m = \frac{1}{2}$ kg

$2k = 100$ N so $k = 50$ N/m

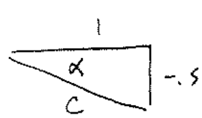
$$\frac{1}{2}x'' + 50x = 0$$

$$\begin{cases} x'' + 100x = 0 \\ x_0 = 1 \text{ m} \\ V_0 = -5 \text{ m/s} \end{cases}$$

$x_H(t) = A\cos 10t + B\sin 10t$

use IV's to find A & B:

ans.: $x_H(t) = \cos 10t \quad -.5\sin 10t$



$C = \sqrt{1.25} \approx 1.118$

$\alpha = \arctan(-.5) \approx -.464$

$x_H(t) \approx 1.118\cos(10t + .464)$ (our ans) $= 1.118\cos(10(t + .0464))$

$\approx 1.118\cos(10t - 5.8195)$ (book ans)

These are the same $x_H(t)$!

Math 2270-1
Simple harmonic motion
Example 1 page 187

```
> x:=t->cos(10*t)-.5*sin(10*t);
```

$$x := t \rightarrow \cos(10\,t) - 0.5\,\sin(10\,t)$$

```
> C:=sqrt(1.25);  #amplitude
  alpha:=arctan(-.5);  #phase
  alpha+evalf(2*Pi); #book's phase
  delta:=alpha/10; #lag
```

$$C := 1.118033989$$

$$\alpha := -0.4636476090$$

$$5.819537699$$

$$\delta := -0.04636476090$$

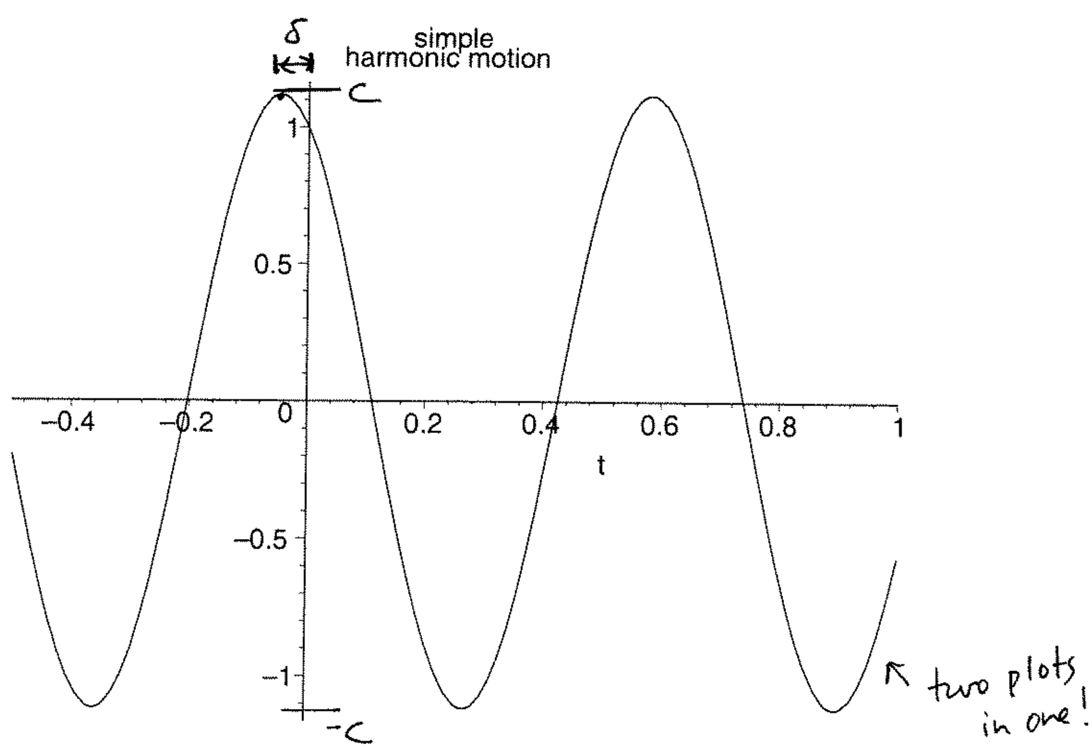
```
> y:=t->C*cos(10*t-alpha); #same function!
  omega:=10:  #angular frequency
  nu:=evalf(10/(2*Pi));  #frequency
  T:=1/nu;  #period
>
```

$$y := t \rightarrow C\,\cos(10\,t - \alpha)$$

$$\nu := 1.591549430$$

$$T := 0.6283185311$$

```
> with(plots):
  plot({x(t),y(t)},t=-.5..1,color=black,title=`simple
  harmonic motion`);
```




```
>
```