Math 2280-1
Mon Feb 6

Recall from Friday:

Theorem 1: If the characteristic polynomial $p(r)$ of the n^{th} order const coeff linear operator L has n distinct roots $r_1, r_2, \ldots r_n$ the a basis for $\ker(L)$ is

$$\{e^{r_1 x}, e^{r_2 x}, \ldots e^{r_n x}\}$$

i.e. $y_H = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$

- redo the example on page 4 Fri the sophisticated way, writing $L(y) = y'' - 2y' + y = (D-I)\circ(D-I)y$ where $D(y) = \frac{dy}{dx}$, $I(y) = y$.

Theorem 2 Let $L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y$ a_j const

So charact poly

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

and $L = D^{(n)} + a_{n-1} D^{(n-1)} + \ldots + a_1 D + a_0$ (where D^k means $D\circ D\circ\ldots\circ D$ k ~~two~~ times).

If $p(r) = (r-r_1)^{k_1} q(r)$ (degree $q_1 = n - k_1$)

Then $\{e^{r_1 x}, x e^{r_1 x}, \ldots x^{k-1} e^{r_1 x}\}$ are k lin. ind solutions to $L(y) = 0$.

proof: $L(y) = p(D)(y)$

$$= q(D)\circ(D-r_1 I)^k (y)$$

$$= q(D)\circ\left[(D-r_1 I)^k (y)\right]$$

So $L(x^m e^{r_1 x}) = q(D)\circ\left[(D-r_1 I)^k (x^m e^{r_1 x})\right]$

$= 0$ for $m \le k-1$

$\therefore$ $\frac{1}{2}$ ■

$(D-r_1 I)(e^{r_1 x}) = r_1 e^{r_1 x} - r_1 e^{r_1 x} = 0$

$(D-r_1 I)(x e^{r_1 x}) = e^{r_1 x} + r_1 x e^{r_1 x} - r_1 x e^{r_1 x}$
$= e^{r_1 x}$; $(D-r_1 I)^2 (x e^{r_1 x}) = 0.$

$(D-r_1 I)\, x^m e^{r_1 x} = m x^{m-1} e^{r_1 x}$

one lower power

inductively,

$$(D-r_1 I)^{m+1}(x^m e^{r_1 x}) = 0$$

Notice $\{e^{r_1 x}, xe^{r_1 x}, \ldots x^{k-1} e^{r_1 x}\}$ are lin ind on any interval, since

$$c_1 e^{r_1 x} + c_2 x e^{r_1 x} + \cdots + c_{k-1} x^{k-1} e^{r_1 x} \equiv 0$$

iff $\quad e^{r_1 x}\left[c_1 + c_2 x + \cdots + c_{k-1} x^{k-1}\right] \equiv 0$

iff $\quad c_1 + c_2 x + \cdots + c_{k-1} x^{k-1} \equiv 0$

iff $\quad c_1 = c_2 = \cdots = c_{k-1} = 0$ since $\{1, x, x^2, \ldots x^{k-1}\}$ are lin ind.

∎

<u>Theorem 3</u> If $p(r)$ factors,

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \cdots (r - r_\ell)^{k_\ell} \qquad r_1, r_2, \ldots r_\ell \text{ distinct}$$

then a basis for $L(y) = 0$ sol'ns is

$$\left\{ e^{r_1 x}, xe^{r_1 x}, \ldots x^{k_1 - 1} e^{r_1 x}, e^{r_2 x}, xe^{r_2 x}, \ldots x^{k_2 - 1} e^{r_2 x}, \ldots \right\}.$$

<u>proof</u>: From thm 2, the y_j all satisfy $L(y_j) = 0$.
It is more work to show they are linearly independent (but true!)

<u>example</u>: Find the general sol'n to

$$y^{(4)} - 2y^{(2)} + y = 0$$

<u>Theorem 4</u> If r is a complex root of $p(r)$, $r = a+bi$,

then $y_1 = e^{ax}\cos bx$
$y_2 = e^{ax}\sin bx$
are to lin ind. sol'ns to $L(y)=0$.

<u>proof</u>: Recall Euler's formula:

$e^{i\theta} := \cos\theta + i\sin\theta$ (Taylor series agree!)

leads to

$e^{\alpha+i\beta} := e^{\alpha}\overbrace{(\cos\beta + i\sin\beta)}^{e^{i\beta}}$ if α, β real

so $e^{(a+ib)x} := e^{ax}(\cos bx + i\sin bx)$ (x real)

check

$\frac{d}{dx} e^{(a+ib)x} = (a+ib)e^{(a+ib)x}$: (Note, we <u>define</u> $\frac{d}{dx}\left(f(x)+ig(x)\right) := f'(x)+ig'(x)$ for f, g real fcns)

Thus if $r = a+ib$ and $y = e^{rx}$

then $L(y) = \underset{\substack{\uparrow \\ \text{charact poly}}}{p(r)}\, e^{rx}$ just as before

so if $a+ib$ is a root of $p(r)$, then $L(y)=0$!

but $L\left(e^{(a+bi)x}\right) = 0 = 0 + 0i$

$$\begin{aligned} &\parallel \\ &L\left(e^{ax}\cos bx + i e^{ax}\sin bx\right) \\ &\parallel \\ &\underbrace{L\left(e^{ax}\cos bx\right)}_{\text{real}} + \underbrace{i\, L\left(e^{ax}\sin bx\right)}_{\text{imag}} \equiv 0 + 0i \end{aligned}$$

(because we're assuming L has real coefficients)

So $L(e^{ax}\cos bx) = 0$
& $L(e^{ax}\sin bx) = 0$!

∎

Example :

Find the general sol'n to

$$y'' + 4y' + 5y = 0$$

Theorem 4 : Can you guess the additional soltns to $L(y)=0$ if $r=a+bi$ is a root of multiplicity k in $p(r)$?

Can you describe a basis for $\ker(L)$ in general?