Math 2280-1
Fri 3 Feb.

HW for Fri 2/10
3.1 1 (2, 4) 5 (11, 13, 17) 27, 29, 30, 31, 33, (34) (35)
3.2 (2) 5 (9) 11 (13) 21 (22) 25 (26)
3.3 (3) (10) (14) 21, (22) (29) (33) (37)
3.4 (4) (5) (6) 13 (15) (19) (23)

Recall,

$$L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$$

$y \in C^n(I) = V$
$a_j(x) \in C(I) \quad j = 0, 1, \dots n-1$
$L: V \to W = C(I)$ is linear

So, the general soln to

$$L(y) = f \qquad (f \in C(I))$$

can be written as $y_P + y_H$
where y_P is any fixed (particular) soln to $L(y) = f$
and $y_H \in \ker(L)$, i.e. $L(y_H) = 0$

And from $\exists!$ thm for IVP, deduced $\dim(\ker L) = n$.

If $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y$ with $a_0, a_1, \dots a_{n-1}$ constants we say L is linear with constant coefficients

In this case we always first attempt to find a basis for $\ker L$ made out of exponentials:

try $y = e^{rx}$

$$\Rightarrow \quad L(y) = \underbrace{(r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0)}_{p(r)} e^{rx} \equiv 0 \quad \boxed{\text{iff } p(r) = 0}$$

$p(r)$ is called the characteristic polynomial of L. (we'll connect this 2270 notion of characteristic polynomial in chapter 5)

Example :

(1) Find all sol'ns to $y''' - y'' - 2y' = 0$

ans: $y_H(x) = c_1 + c_2 e^{2x} + c_3 e^{-x}$

(2) Find a particular sol'n, then all sol'ns, to

$$y''' - y'' - 2y' = 2e^x$$

we'll discuss how to do this systematically in §3.5 - for now let's try to "guess",

ans: ?

(3) Solve the IVP:

$$\begin{cases} y''' - y'' - 2y' = x + 1 \\ y(0) = 0 \\ y'(0) = 2 \\ y''(0) = 2 \end{cases}$$

ans: $y = -e^x + 1 + e^{2x} - e^{-x}$

Theorem 1 If the characteristic polynomial $p(r)$ (of the n^{th} order const coeff linear operator L) has n distinct roots $r_1, r_2, \dots r_n$ (i.e. $i \neq j \Rightarrow r_i \neq r_j$)
then a basis of $\ker(L)$ is given by

$$\{ e^{r_1 x}, e^{r_2 x}, \dots e^{r_n x} \}$$

i.e.

$$y_H = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

proof: since $\dim(\ker L) = n$ we need only show linear independence.

Recall our Wronskian discussion Wednesday;
$\{y_1, y_2, \dots y_n\}$ are linearly independent
if the Wronskian matrix

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ y_1'' & \vdots & & \vdots \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}$$

is invertible for some x_0

i.e., if its det, called the Wronskian $W \neq 0$.

For $y_1 = e^{r_1 x}, y_2 e^{r_2 x}, \dots y_n e^{r_n x}$
the Wronskian matrix at $x = 0$ is the Vandermond matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & & r_n \\ r_1^2 & r_2^2 & & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & & r_n^{n-1} \end{bmatrix}$$

so $W = \prod_{i<j} (r_j - r_i) \neq 0$ ■

(this was a neat proof by induction in Math 2270, see e.g. Bretscher §6.2 #31 hw)

Remark: a neat fact is that if $\{y_1, y_2, \dots y_n\}$ all satisfy the n^{th} order DE $L(y) = 0$ where L is the operator on page 1, $L(y) = y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots$
(↑ coeff of 1, i.e. original $a_n(x) \neq 0$ on I.)
Then $\{y_1, y_2, \dots y_n\}$ are linearly independent if and only if $W \neq 0 \ \forall x \in I$

(proof $\Leftarrow$: $W \neq 0$ for even one $x_0 \in I \Rightarrow \{y_1, \dots y_n\}$ lin ind.
$\Rightarrow$: $\{y_1, \dots y_n\}$ l.i. $\Rightarrow$ basis for $\ker L$ $\Rightarrow$ can solve any IVP (for any $x_1 \in I$)

$$\begin{cases} L(y) = 0 \\ \vec{IV}(y)(x_1) = \vec{b} \end{cases}$$

but $y = c_1 y_1 + \dots + c_n y_n \Rightarrow \vec{IV}(\vec{y}) = \begin{bmatrix} y_1 & y_2 & y_n \\ y_1' & y_2' & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$
$\Rightarrow$ wronksian matrix has rank n at x_1
[illegible])

example: What is the Thm 1 Wronskian at general x?

repeated roots:

<u>example</u>: Find all solns to

$$y'' - 2y' + y = 0$$

$p(r) = r^2 - 2r + 1 = (r-1)^2$; $y_1 = e^x$

$$L(y) = (D^2 - 2D + I)y$$
$$= (D-I)\circ(D-I)y$$

↑ composition.

(but often write $(D-I)^2$)

$W(y_1, y_2) = \det \begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} \overset{@x=0}{=} \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$

so y_1, y_2 are lin ind!

$y_2 = xe^x$ will work!!!

<u>check 1</u>:
$1\,(\quad y_2 = xe^x$
$-2\,(\quad y_2' = xe^x + e^x$
$1\,(\quad y_2'' = xe^x + 2e^x$

$$L(y_2) = e^x\left[x(1-2+1) + 1(2-2)\right] = 0!$$

$$\boxed{y_H(x) = c_1 e^x + c_2 xe^x}$$

<u>sophisticated check</u>

$$(D-I)(xe^x) = (e^x + xe^x - xe^x)$$
$$= e^x$$

so $(D-I)\circ\underbrace{(D-I)e^x}_{e^x} = 0$!

[this will generalize to
$p(r) = (r-r_1)^k \underbrace{q(r)}_{q(r_1)\neq 0}$
then $\{e^{r_1 x}, xe^{r_1 x}, \ldots x^{k-1}e^{r_1 x}\}$ will all solve $L(y) = 0$]