Math 2280-1
Tues 2/14

Review, after we discuss

variation of parameters : If, for $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$
you already have a basis for ker(L), i.e. the sol'n set to $L(y)=0$,
there is a formula to find a particular sol'tn y_p to

$$L(y_p) = f.$$

derivation: Let $\{y_1, y_2, \dots y_n\}$ a basis for ker(L)

we search for y_p expressed as

$$
\begin{aligned}
a_0(\quad & y_p(x) = u_1 y_1 + u_2 y_2 + \dots + u_n y_n \qquad u_j = u_j(x) \text{ fcns! (hence the name of the method).}\\
a_1(\quad & y_p' = u_1 y_1' + u_2 y_2' + \dots + u_n y_n' + (u_1' y_1 + u_2' y_2 + \dots + u_n' y_n) \rightarrow \text{set} \equiv 0\\
+\, a_2(\quad & y_p'' = u_1 y_1'' + u_2 y_2'' + \dots + u_n y_n'' + (u_1' y_1' + u_2' y_2' + \dots + u_n' y_n') \rightarrow \text{set} \equiv 0\\
& \vdots \\
+\, a_{n-1}(\quad & \\
+\, 1(\quad & y_p^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \dots + u_n y_n^{(n)} + (u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)}) \rightarrow \text{set} \equiv f
\end{aligned}
$$

$$\Longrightarrow \quad L(y_p) = \underbrace{u_1 L(y_1) + \dots + u_n L(y_n)}_{=0!} + f$$

conditions in matrix form

$$\underbrace{\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}}_{W(y_1,\dots,y_n).} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

$$\boxed{\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}}$$

then find choices of each $u_j(x)$ by antidifferentiation,

& get
$y_p = u_1 y_1 + \dots + u_n y_n$!

since $\{y_1, \dots y_n\}$ is a basis we can solve all IVP's at each $x \Rightarrow [W]$ has rank n at each $x \Rightarrow W^{-1}$ exists at each x!

Example

forced spring, $mx'' + cx' + kx = F(t)$; e.g.

$$x''(t) + 16x(t) = \sin 4t$$

what would your guess be for $x_p(t)$ using guessing?

$$L(x) := x'' + 16x$$

$$\ker L = \operatorname{span}\{\underset{\underset{x_1}{\uparrow}}{\cos 4t},\ \underset{\underset{x_2}{\uparrow}}{\sin 4t}\}.$$

var par formula from page 1:

$$\begin{bmatrix} \cos 4t & \sin 4t \\ -4\sin 4t & 4\cos 4t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin 4t \end{bmatrix}.$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4\cos 4t & -\sin 4t \\ 4\sin 4t & \cos 4t \end{bmatrix}\begin{bmatrix} 0 \\ \sin 4t \end{bmatrix} = \begin{bmatrix} -\sin^2 4t \\ \sin 4t\cos 4t \end{bmatrix}$$

$$u_1' = -\sin^2 4t = \frac{\cos 8t - 1}{2} \qquad \text{(double/half angle formula)}$$

$$u_2' = \sin 4t \cos 4t$$

choose $u_1 = \dfrac{\sin 8t}{16} - \dfrac{t}{2}$

$u_2 = \dfrac{1}{8}\sin^2 4t$

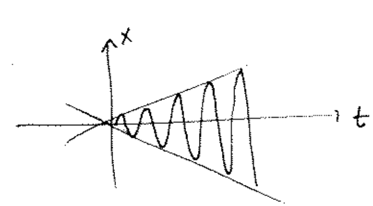
$$x_p = u_1x_1 + u_2x_2 = \left(\frac{\sin 8t}{16} - \frac{t}{2}\right)(\cos 4t) + \frac{1}{8}\sin^2 4t\,(\sin 4t)$$

$$= \frac{2}{16}\sin 4t\,\cancel{\cos 4t}\,\cos 4t - \frac{t}{2}\cos 4t + \frac{1}{8}(1-\cancel{\cos^2 4t})(\sin 4t)$$

$$= -\frac{t}{2}\cos 4t + \underbrace{\frac{1}{8}\sin 4t}_{\text{solves } L(x)=0}$$

so may take

$$\boxed{x_p(t) = -\frac{t}{2}\cos 4t}$$



resonance!