Math 2280-1
Monday Feb. 13

§3.5 (not on Wed. exam).

We are now experts at finding the general homogeneous sol'n to n^{th} order const. coeff DE's, i.e.

for

$$L = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y \qquad \text{with } a_j = \text{const}, \ j = 0, 1, \ldots n-1$$

We can always construct a basis for the n-dim'l space of sol'ns to

$$L(y) = 0.$$

§3.5 contains algorithms for finding particular sol'ns for the non-homogeneous DE,

$$L(y_P) = f$$

(and then the full sol'n will be $y = y_P + y_H$)

Method 1 "Method of undetermined coeffs" (i.e. guessing)
(Actually depends on linear algebra)

example 1 : $y' + 4y = e^x$

$y_H = Ce^{-4x}$

try $y_P = Ce^x$ since $L(e^{rx}) = p(r)e^{rx}$
so for $V = \text{span}\{e^x\}$,
$L: V \to V$.

$$\begin{array}{rl} 4(& y_P = Ce^x \\ 1(& y_P' = Ce^x \\ \hline & L(y_P) = 5Ce^x \overset{\text{want}}{=} e^x \end{array}$$

$C = 1/5$

$$y = y_P + y_H = \frac{1}{5}e^x + Ce^{-4x}$$

example 2 $y' + 4y = \cos 2x$

for $V = \text{span}\{\cos 2x, \sin 2x\}$ need to include this fcn!

$L : V \to V$

on $\underline{V}$, $\ker L = \{0\}$ (since on $C^1(\mathbb{R})$, $\ker L = \text{span}\{e^{-4x}\}$)

so $\exists!\ v \in V$ s.t. $Lv = \cos 2x$ (rank + nullity thm!).

book's way.

$$\begin{aligned} 4(\quad & \text{try } y_p = A\cos 2x + B\sin 2x \\ 1(\quad & y_p' = -2A\sin 2x + 2B\cos 2x \end{aligned}$$

$$L(y_p) = \cos 2x\,[4A + 2B] + \sin 2x\,[-2A + 4B] \overset{\text{want}}{=} \cos 2x\,[1] + \sin 2x\,[0]$$

so we want to solve this eqtn for the coords of the sol'n!

$$\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/10 \end{bmatrix}$$

$$\boxed{y_p = \frac{1}{5}\cos 2x + \frac{1}{10}\sin 2x}$$

clever 2270 graduate's way. (equivalent, but quicker!)

basis $\mathcal{B} = \{\underset{f_1}{\cos 2x}, \underset{f_2}{\sin 2x}\}$

matrix for L wrt $\mathcal{B}$ $\to$ $[L]_{\mathcal{B}} = \left[\, [L(f_1)]_{\mathcal{B}} \,\middle|\, [Lf_2]_{\mathcal{B}} \,\right]$

$$= \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$$

$\uparrow$ $L(\cos 2x) = 4\cos 2x - 2\sin 2x$ $\qquad$ $\uparrow$ $[L(\sin 2x)]_{\mathcal{B}}$

example 3: find y_p for $L(y_p) = x + 2$

hint: let $V = P_1 = \text{span}\{1, x\}$.

ans: $y_p = \frac{1}{4}x + \frac{7}{16}$

Example 4 Using previous 3 examples, find the full sol'n to

$$y' + 4y = 2e^x - 3\cos x - 5x - 10$$

hint: superposition.

See guessing table page 202

$f(x)$ (or a piece of f)	guess
e^{rx}	Ce^{rx}
$e^{ax}(A\cos bx + B\sin bx)$	$e^{ax}(D\cos bx + E\sin bx)$
$P_n(x)$	$Q_n(x)$
$P_m(x)e^{rx}$	$Q_m(x)e^{rx}$
etc	

$V = \text{span}\{e^{rx}\}$ $L: V \to V$
L is 1-1 & onto iff r is not a root of charact poly.

$V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx\}$
L is 1-1 & onto iff $a \pm bi$ not roots of char. poly

$V = \mathbb{P}_n = \text{span}\{1, x, \dots x^n\}$
L is 1-1 & onto iff 0 is not a root of c.p.

$V = \text{span}\{e^{rx}, xe^{rx}, \dots x^m e^{rx}\}$
works if r is not a root of charact poly.

Fixit recipe if L has non-trivial kernel for original V:

If at least one term in guess solves homogeneous eq'n, multiply entire guess by x^s where s is the smallest counting # s.t. no term in new guess solves homogeneous DE !!

Example

$$L(y) = y'' + 2y' + y$$

solve

$$L(y) = e^{-x}$$

notice $p(r) = (r^2 + 2r + 1) = (r+1)^2$

ans: $y_p = \frac{1}{2}x^2 e^{-x}$!

Review sheet for exam 1.

1^{st} order ODE's. Chapters 1.1–1.5, 2.1–2.3 (not 1.6, 2.4–2.6)

- Recognize and solve DE's and IVP's for

  $\frac{dy}{dx} = f(x)$ §1.2 antidiff

  $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ §1.4 separable

  $\frac{dy}{dx} + P(x)y = Q(x)$ §1.5 linear

- slope fields, $\exists!$ for IVP §1.3, 2.2
  phase portraits.
  - geometric meaning of $\frac{dy}{dx} = f(x,y)$ for the graph $y = y(x)$
  - sketch slope fields using isoclines
  - $\exists!$ thm for IVP
  - autonomous DE's $\frac{dy}{dx} = f(y)$
    - equilibrium sol'ns
    - phase portraits
    - stability

- applications & modeling
  - separable
    - exp growth & decay
    - Newton's law of cooling
    - Toricelli
    - logistic / doomsday ext / harvesting (§2.1)
    - acceleration with drag terms (§2.3)
  - linear
    - mainly mixing problems (§1.5)
    - also some of §2.3 models
    - if $P(x)$ is const, i.e.
      $y' + a_0 y = f$
      this theory connects to chapter 3 $y = y_P + y_H \cdots$ use of e^{rt}, etc.

Higher order linear DE's (3.1–3.4)

- Why $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$ is called linear
  - Why the general sol'n to $L(y) = f$ is $y = y_P + y_H$; more general superposition
  - Dimension of sol'n space to $L(y) = 0$ (and its relationship to $\exists!$ thm for IVP)
  - linear ind & dep of fns
    - Wronskian matrix & determinant

- Solving $L(y) = 0$ if L has constant coeffs a_j
  e^{rx}, $p(r)$, real roots (distinct, repeated), complex roots (distinct & repeated)
  using complex exponentials, Euler, cos & sin addition angle formulas

- Mechanical vibrations
  - pendulum & spring models
  - simple harmonic motion (amp, phase, ang. freq, freq, period, etc. ABC triangle)
  - the three kinds of damping, and corresponding solution types.