Math 2280-1
Wed Feb 1
§3.1-3.2

Part of next Friday's (Feb 10)
assignment (if you wish
to stay current with the lectures)
3.1 1 2, 4 5 11, 13, 17 27, 29, 30, 31, 33, 34 35
3.2 2 5 9 11 13 21 22 25 26

Recall from Tues:

$$L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$$

defined for $y \in V := \{ y : I \to \mathbb{R},\ y(x) \text{ is } n\text{-times continuously differentiable} \}$
(I is an interval in $\mathbb{R}$)
$:= C^n(I)$
each $a_j(x) \in C^0(I) = C(I)$

- Then $L : V \to \underset{C^0(I)}{\overset{\shortparallel}{W}}$ is linear transformation.
(we checked this)

- If $x_0 \in I$ and $\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} := \vec{b} \in \mathbb{R}^n$ and $f \in C(I)$

then the initial value problem

$$\text{IVP} \quad \begin{cases} L(y) = f \\ \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} = \vec{b} \end{cases}$$

has a unique solution on I

(we checked this for $n=1$, spring intuition ($n=2$),
Taylor series heuristic reasoning general n)

- Every solution to
$L(y) = f$
is of the form $y = y_P + y_H$, where y_P is a particular solution
and $y_H \in \ker(L)$ (i.e. $L(y_H) = 0$)
note: our text writes y_C for y_H, and calls it the complementary sol'n
(this is a vector space theorem; we proved it.)

Our strategy for n^{th} order linear DE's will be to
find a particular solution and a basis for $\ker(L)$ (i.e. all homogeneous sol'ns will be linear combos of the basis)

in order to find all sol'ns to
$L(y) = f$

Theorem: Let $L(y)$ be the n^{th} order linear operator defined on page 1.

Then the dimension of $\ker(L) = n$.

(so if we can find n linearly ind. sol'ns to $L(y)=0$, they will be a basis for $\ker(L)$, i.e. every sol'n will be a linear combo of the basis sol'ns,

$$y_H = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \,)$$

proof:

If $y(x) \in V$

write $\vec{IV}(y) := \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}$

By the $\exists!$ theorem for IVP, $\exists$ sol'ns $y_1, y_2, \dots y_n$ to $L(y)=0$

such that

$$\vec{IV}(y_j) = \vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{entry}_j$$

Show $\{y_1, \dots y_n\}$ are a basis:

- These sol'ns are <u>linearly independent</u>, because if

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n \equiv 0 \qquad (\forall x \in I)$$

$$\frac{d}{dx} \Rightarrow \quad c_1 y_1' + c_2 y_2' + \dots + c_n y_n' \equiv 0$$

$$\frac{d^2}{dx^2} \Rightarrow \quad c_1 y_1'' + c_2 y_2'' + \dots + c_n y_n'' \equiv 0$$

$$\vdots$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} \equiv 0$$

called the Wronskian matrix of $y_1 \dots y_n$; its det is called Wronskian

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \qquad \forall x \in I$$

shows linear independence.

In particular, at $x = x_0$, the Wronskian matrix (above) is the identity, so $\vec{c} = \vec{0}$ ■

- These solutions <u>span</u> $\ker(L)$ because if $z(x)$ solves $L(z)=0$, then

for $\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} := \vec{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$, the (other?) sol'n $y(x) = b_0 y_1(x) + b_1 y_2(x) + \dots + b_{n-1} y_n(x)$

has the same initial value vector!, i.e. $\vec{IV}(y) = \vec{b}$.

So, by uniqueness (in $\exists!$ theorem), $z \equiv y$! ■

<u>example</u> : Let $L(y) = y'' + y' - 30y$

Find the $\ker(L)$, i.e. the general sol'n to $y''+y'-30y=0$

hint: to find a potential basis try $y = e^{rx}$

<u>Show</u> linear ind. of you 2 sol'ns y_1, y_2 to show they are a basis.
(Hint: you could show the Wronskian (det) is non-zero)

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motivation to try $y=e^{rx}$ for any $\underbrace{\text{constant coefficient}}_{a_j(x) \equiv a_j \text{ const}}$ linear homogeneous DE:

In the example above,
if we write $D(y) := \frac{dy}{dx}$
$I(y) := y$ (identity)

then the L above can be written as

$$L = D^2 + D - 30I = (D+6I)(D-5I) = (D-5I)(D+6I)$$

["multiplication" of operators means composition here!]

and e^{-6x} solves $(D+6I)y=0$
e^{5x} solves $(D-5I)y=0$

this principle holds in general

Checking for linear independence (using the Wronskian)

- Show $\{1, x, x^2\}$ are linearly ind. fcns on any interval.

- Show $\{e^x, e^{-x}, e^{2x}\}$ are linearly ind fcns on any interval.

- Show that if $r_1, r_2, \ldots r_n$ are distinct, then
  $\{e^{r_1 x}, e^{r_2 x}, \ldots e^{r_n x}\}$ are linearly independent.