Name.......SOLUTIONS.......
I.D. number.......

Math 2280-1
Second Midterm
April 4, 2006

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Let A be the matrix

$$A := \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$$

1a) Find the eigenvalues and eigenvectors of A. (Hint: you should get 0 and -3 for the eigenvalues.) (10 points)

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2+3\lambda+2-2 = \lambda^2+3\lambda = \lambda(\lambda+3)$$
$$=0 \text{ for } \boxed{\lambda = 0, -3}$$

$\lambda=0$:

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\lambda=0 \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\lambda=-3$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \\ \hline 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\lambda=-3 \quad \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

1b) Find the general solution to

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(5 points)

$e^{\lambda t}\vec{v}$ basis

$$\boxed{\vec{x}_H(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

1c) Find all solutions to the nonhomogeneous system of differential equations

$$* \qquad \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

(10 points)

(Hint: try a constant vector for a particular solution.)

$\vec{x}_p = \vec{k}$

substitute into $*$:

$$0 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} \quad ; \quad A\vec{k} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 2 & -4 \\ 1 & -1 & 2 \\ \hline 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$k_2 = s$
$k_1 = 2 + s$

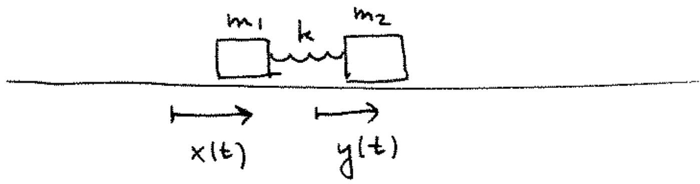
$\vec{k} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} +1 \\ 1 \end{bmatrix}$; you get an $\vec{x}_p$ for any choice of s, e.g. $s = 0$ yields $\vec{k} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

So

$\vec{x}(t) = \vec{x}_p + \vec{x}_H$

$$\boxed{\vec{x}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

2) Consider a mass-spring system consisting of two masses coupled together with one spring and sliding along a frictionless plane, as indicated in the sketch below.



2a) Derive the second order system of differential equations which governs the motion of this system, for the two displacement functions x(t), y(t).

(4 points)

$$m_1 x'' = k(y-x)$$
$$m_2 y'' = -k(y-x)$$

2b) What is the dimension of the solution space to this sytem of differential equations?

(2 points)

$\boxed{\dim = 4}$ since this system is equivalent to a homog. linear ~~strada homog. linear~~ system of 4 1^{st} order DE's,

$$\left(\text{i.e.} \quad \begin{aligned} x_1 &= x \\ x_2 &= x' \\ y_1 &= y \\ y_2 &= y' \end{aligned} \qquad \begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ +\frac{k}{m_1}(y_1 - x_1) \\ y_2 \\ -\frac{k}{m_2}(y_1 - x_1) \end{bmatrix} \right)$$

2c) Suppose the spring constant is 4 Newtons/meter, that $m_1 = 2$ kg and that $m_2 = 4$ kg. Show that in this case your general system from 2a) can be rewritten as

(4 points)

$$\begin{bmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$\Updownarrow$

substitute into 2a):

$$\left.\begin{aligned} 2x'' &= 4(y-x) \\ 4y'' &= -4(y-x) \end{aligned}\right\} \Leftrightarrow \begin{aligned} x'' &= -2x + 2y \\ y'' &= x - y \end{aligned}$$

2d) Find the general solution to the system in (2c). Notice that the matrix is exactly the one you used in problem (1), so you already know its eigenvalues and eigenvectors from (1a). (Hint: For the case $\omega = 0$ you may need to think about the train motion in order to find a second linearly independent solution!)

(10 points)

$\lambda = 0$ $\quad\quad\quad\quad$ $\lambda = -3$

$\omega = \sqrt{-\lambda} = 0$ $\quad\quad$ $\omega = \sqrt{3}$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad\quad$ $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$

$$\vec{x}_H(t) = \underbrace{(c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{moving at a constant speed, in phase}} + \underbrace{(c_3 \cos\sqrt{3}t + c_4 \sin\sqrt{3}t)\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{oscillating out of phase}}$$

$(\cos 0t = 1)$

note, if $A\vec{v} = \vec{0}$

and $\vec{x}(t) = (c_1 + c_2 t)\vec{v}$

then $\vec{x}''(t) = 0 = A(c_1 + c_2 t)\vec{v}$

since $A\vec{v} = \vec{0}$

3) For the matrix

$$B := \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Maple says

```
> eigenvectors(B);
```

$$[2+I, 1, \{[1, I]\}], [2-I, 1, \{[1, -I]\}]$$

Use this information to find a basis of real vector-valued functions, for the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(10 points)

$$e^{(2+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{2t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\, e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$\mathrm{Re}\, e^{\lambda t} \vec{v}$, $\mathrm{Im}\, e^{\lambda t} \vec{v}$ are a basis, i.e.

$$\left\{ e^{2t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \right\}$$

4) Find the matrix exponential $\mathbf{e}^{(C\,t)}$, for

$$C := \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

(Hint: you will either need to use chains or the fact that C=2I+N.)

(10 points)

$C = 2I + N, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$e^{(2I+N)t} = e^{2It} e^{Nt}$ ($e^{A+B} = e^A e^B$ when A & B commute)

$= (e^{2t} I)(I + Nt + \underset{\uparrow}{0})$

all higher order terms in power series are zero because $N^2 = 0$

$= e^{2t}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

$$\boxed{e^{Ct} = \begin{bmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{bmatrix}}$$

---

alternate: $|C - \lambda I| = (\lambda - 2)^2$.

$\lambda = 2$ evect is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

FMS: $\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$; since $\Phi(0) = I$,

$e^{At} = \Phi(t)\Phi(0)^{-1}$
$= \Phi(t)$!

for other basis sol'n construct chain:

$[C - 2I]^2 = N^2 = 0$ so

let $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{u} = [C-2I]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e^{2t}(t\vec{u} + \vec{v}) = e^{2t}\begin{bmatrix} t \\ 1 \end{bmatrix}$

5) Consider the system of differential equations below which model two populations x(t), y(t):

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 8x - x^2 - xy \\ 20y - y^2 - 4xy \end{bmatrix}.$$

5a) Would you call this a predator-prey type problem, a competing species problem, or something else? Explain. (4 points)

competing species, since each population is logistic in the absence of the other, and the presence of either inhibits the other (since each xy coef. is negative).

5b) Find all four equilibrium solutions to this autonomous system. (Hint: You should get [4,4] as one solution.) (6 points)

$x(8-x-y)=0$
$y(20-y-4x)=0$

$x=0$: $y=0$ gives $\begin{bmatrix}0\\0\end{bmatrix}$; $20-y=0$ gives $\begin{bmatrix}0\\20\end{bmatrix}$

$x\neq 0$: $y=0$ gives $\begin{bmatrix}8\\0\end{bmatrix}$; $y\neq 0$ gives $\begin{bmatrix}4\\4\end{bmatrix}$

$x+y=8$
$4x+y=20$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 4 & 1 & 20 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -3 & -12 \end{array}\right] \; -4R_1+R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 4 \end{array}\right]$$

5c) Find the linearized system of differential equations near the equilibrium point [4,4]. Classify what kind of equilibrium this point is. Sketch what the phase portrait looks like near this particular equilibrium, accurately using eigenvalue and eigenvector information from the Jacobian matrix. (15 points)

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 8-2x-y & -x \\ -4y & 20-2y-4x \end{bmatrix}$$

@ $\begin{bmatrix}4\\4\end{bmatrix}$, $J = \begin{bmatrix} -4 & -4 \\ -16 & -4 \end{bmatrix}$

$$\boxed{\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -16 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}$$

where $x = 4+u$
$y = 4+v$

and higher order terms in exact system for $\begin{bmatrix}u'\\v'\end{bmatrix}$ are ignored.

$$\begin{vmatrix} -4-\lambda & -4 \\ -16 & -4-\lambda \end{vmatrix}$$
$= (\lambda+4)^2 - 64$
$= (\lambda+4+8)(\lambda+4-8)$
$= (\lambda+12)(\lambda-4)$

$\lambda = 4$

$$\left[\begin{array}{cc|c} -8 & -4 & 0 \\ -16 & -8 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

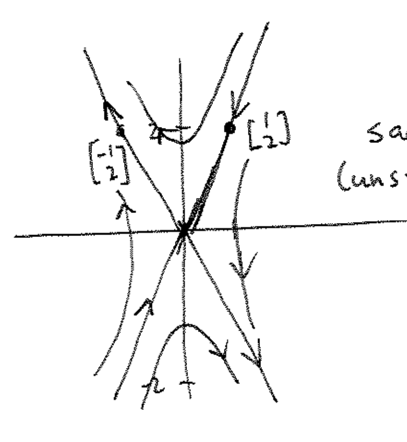
$\vec{v} = \begin{bmatrix}-1\\2\end{bmatrix}$

$\lambda = -12$

$$\left[\begin{array}{cc|c} 8 & -4 & 0 \\ -16 & 8 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$

$$\vec{u}(t) = c_1 e^{4t}\begin{bmatrix}-1\\2\end{bmatrix} + c_2 e^{-12t}\begin{bmatrix}1\\2\end{bmatrix}$$



saddle (unstable)

5d) As you may perhaps discern from the picture below, the three equilibria other than [4,4] are nodes (two are stable sinks and one is an unstable source). Fill in the missing rectangle of the phase portrait below (using 5c, and noticing that the x and y directions are scaled differently in this picture), for the nonlinear population model we are considering. Discuss the implications for solutions to the initial value problem when both initial populations are positive. Your discussion will be aided by a sketch of several key trajectories which you add onto the portrait.

This was the competition model with $c_1c_2 > b_1b_2$, so one species goes extinct: (10 points)

I drew in the separatrices for the saddle at $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$

in unshaded region, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} 0 \\ 20 \end{bmatrix}$ ($x(t)$ dies out)

in shaded region, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ ($y(t)$ dies out)