Name: SOLUTIONS

I.D. number

Math 2280-1

Exam 1

February 15, 2006

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin.
Good Luck!

1) Consider a mass-spring system with damping, in which the mass is 3 kg.
1a) What values of damping coefficient and spring constant (including correct units!) would lead to the following differential equation for the displacement x(t) from equilibrium?

$$3\,(x''(t) + 4x'(t) + 3x(t) = 0$$

$$3x'' + cx' + kx = 0$$

(4 points)

$$\Rightarrow c = 12 \quad Ns/m \quad (\text{or } kg/s)$$
$$k = 9 \quad N/m$$

[deduce units because mx'' has units $kg\,m/s^2 = N$, as do all 3 terms]

1b) Find the general solution to the differential equation in 1a. What kind of damping is exhibited by this spring system?

(10 points)

$$x'' + 4x' + 3x = 0$$

$$x = e^{rt} \rightarrow p(r) = r^2 + 4r + 3$$
$$= (r+3)(r+1)$$

$$r = -1, -3$$

$$x_H(t) = c_1 e^{-3t} + c_2 e^{-t}$$

$\boxed{\cancel{\text{under}}\ \text{over!! damped}}$

1c) Solve the initial value problem for the spring system in 1a, with x(0)=0, Dx(0)=2.

(8 points)

$x(t) = c_1 e^{-3t} + c_2 e^{-t}$

$x(0) = 0$

$x'(0) = 2$

$x(0) = c_1 + c_2 = 0$

$x'(0) = -3c_1 - c_2 = 2$

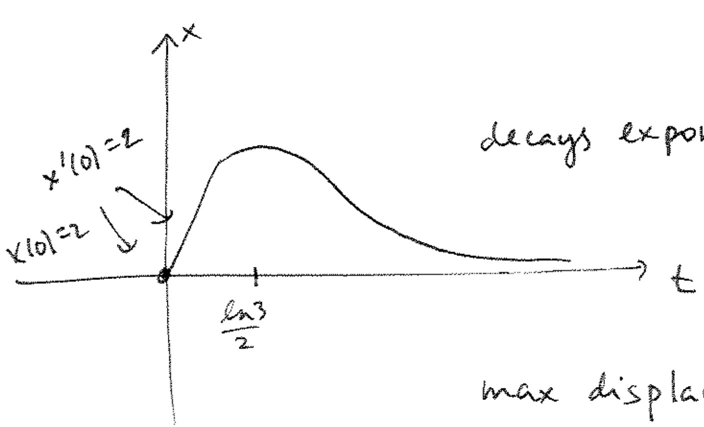
add eqtns $\Rightarrow -2c_1 = 2$ $\boxed{c_1 = -1 \quad c_2 = 1}$

$$\boxed{x(t) = -e^{-3t} + e^{-t}}$$

1d) Sketch a qualitative graph of your solution x(t) to part (c) as a function of t. Solve for the time when the mass is displaced a maximum amount from equilibrium.

(8 points)

$x(t) = e^{-t}\left[1 - e^{-2t}\right]$ only root at $t=0$



decays exponentially.

max displacement when $x'(t) = 0$

$x'(t) = 3e^{-3t} - e^{-t}$

$= -e^{-t}\left[1 - 3e^{-2t}\right]$

$3e^{-2t} = 1$

$e^{-2t} = 1/3$

$-2t = -\ln(3)$

$\boxed{t = \frac{\ln 3}{2}}$

2) Consider the differential equation

$$\frac{dP}{dt} = P^2 - 4P - 5.$$

which could be a model for a certain population.

2a) Find the equilibrium solutions.

(6 points)

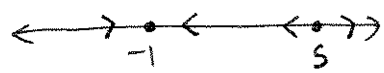
$$\frac{dP}{dt} = (P-5)(P+1)$$

equil sol'ns $P = 5$
$P = -1$

2b) Sketch the phase diagram for this differential equation. Use this phase diagram to deduce the stability of your (2a) equilibrium solutions

(6 points)

phase portrait:



so $P=-1$ is stable and $P=5$ is unstable.

2c) Describe a population model which could lead to differential equation of this type. Be as precise as you can, so that you can account for all three terms on the right-hand side of the differential equation.

(4 points)

$\frac{dP}{dt} = P^2 - 4P = P(P-4)$ is doomsday-extinction

so this could be doomsday-extinction with harvesting (at rate 5 pop units/time unit)

2e) Solve the initial value problem for this differential equation, with P(0)=3. Explain what happens to your solution as time increases, and explain why this is consistent with your phase diagram in (2b).

(14 points)

$$\frac{dP}{(P-5)(P+1)} = dt$$

$$\frac{1}{6}\left(\frac{1}{P-5} - \frac{1}{P+1}\right) dP = dt$$

$$\frac{1}{P-5} - \frac{1}{P+1} \, dP = 6\, dt$$

$$\ln\left|\frac{P-5}{P+1}\right| = 6t + C$$

$$\frac{P-5}{P+1} = C e^{6t}$$

$$P(0)=3 \Rightarrow C = \frac{3-5}{3+1} = -\frac{1}{2}$$

$$(P-5) = -\frac{1}{2} e^{6t} (P+1)$$

$$P\left[1 + \frac{1}{2}e^{6t}\right] = 5 - \frac{1}{2}e^{6t}$$

$$P = \frac{5 - \frac{1}{2}e^{6t}}{1 + \frac{1}{2}e^{6t}} = \frac{10 - e^{6t}}{2 + e^{6t}} \; \frac{e^{-6t}}{e^{-6t}} = \frac{-1 + 10e^{-6t}}{1 + 2e^{-6t}}$$

$P(t) \to -1$ as $t \to \infty$

(so for a real population there was doomsday at finite time)

as predicted by phase portrait

3a) Using the Chapter 1 algorithm for solving first order linear differential equations, find the solution to the initial value problem

$$x'(t) + \frac{3}{50+3t}\, x(t) = 3$$
$$x(0) = 0.$$

(10 points)

$$\underbrace{e^{\int \frac{3}{50+3t}dt}}_{e^{\ln(50+3t)}} \left(x' + \frac{3}{50+3t} x \right) = 3 e^{\int \frac{3}{50+3t}dt}$$

$$e^{\ln(50+3t)} = $$

$$(50+3t)\left(x' + \frac{3}{50+3t} x\right) = 3(50+3t)$$

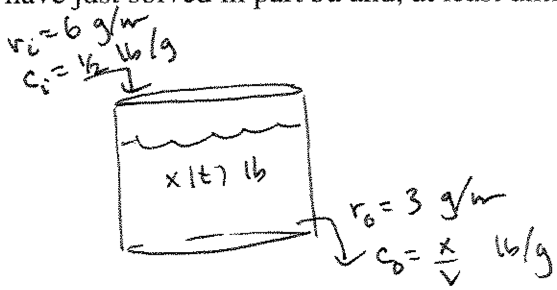
$$[(50+3t)x]' = 3(50+3t)$$

$$(50+3t)x = 150t + \tfrac{9}{2}t^2 + C$$

$$x(0) = 0 \Rightarrow C = 0. \qquad \boxed{x(t) = \frac{150t + \frac{9}{2}t^2}{50+3t}}$$

3b) Consider a mixing tank with one input pipe and ~~one output pipe.~~ Assume the tank initially contains 50 gallons of salt-free water, Assume that water flows into the tank at a rate of 6 gallons per minute, and that the input concentration is 1/2 pound of salt per gallon of water. Assume that well-mixed water flows out of the tank at a rate of 3 gallons per minute. Let x(t) be the amount of salt in the tank at time t. Using the input-output tank model we've studied, show that x(t) satsifies the initial value problem which you have just solved in part 3a and, at least until the tank overflows.

(5 points)



water volume $\quad \frac{dV}{dt} = r_i - r_o$

$$= 6 - 3 = 3$$

$$\begin{cases} \frac{dV}{dt} = 3 \\ V(0) = 50 \end{cases} \Rightarrow \underline{V(t) = 50 + 3t}$$

$$\frac{dx}{dt} = \underset{6}{r_i}\,\underset{\frac{1}{2}}{c_i} - \underset{3}{r_o}\,\underset{x/V}{c_o}$$

$$\frac{dx}{dt} = 3 - \frac{3x}{V(t)} = 3 - \frac{3x(t)}{50+3t}$$

i.e.

$$\boxed{\frac{dx}{dt} + \left(\frac{3}{50+3t}\right) x(t) = 3}$$

4a) What is the dimension of the solution space to the differential equation

$$y'''(x) + 4y'(x) = 0$$

(3 points)

if L is n^{th} order linear DE, $\dim \ker L = n$

so in our case $\boxed{\dim = 3}$

4b) Find a basis for the solution space in 4a.

(10 points)

$y = e^{rt}$

$L(y) = e^{rt} p(r) = e^{rt}(r^3 + 4r)$

$r^3 + 4r = r(r^2+4)$

roots $r = 0 \to y_1(t) = 1$

$r = \pm 2i \to y_2(t) = \cos 2t,\ y_3(t) = \sin 2t$

$\{1, \cos 2t, \sin 2t\}$

4c) Prove the functions you found in you part 4b basis are linearly independent. (You may perhaps like to use the Wronskian or related ideas.)

(7 points)

$$W = \det \begin{vmatrix} 1 & \cos 2t & \sin 2t \\ 0 & -2\sin 2t & 2\cos 2t \\ 0 & -4\cos 2t & -4\sin 2t \end{vmatrix} = |M_{11}| = 8\sin^2 2t + 8\cos^2 2t$$

$$= 8 \neq 0$$

so $\{y_1, y_2, y_3\}$ are linearly independent

[the actual variable was "x" not "t", so define $t := x$ in the above]

5) State and derive the formulas which show how C and α are related to A and B, in the identity

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

$$= C\left[\cos\omega t \cos\alpha + \sin\omega t \sin\alpha\right]$$

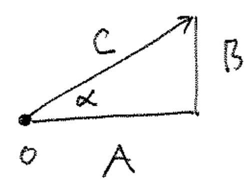
(5 points)

since $\cos\omega t, \sin\omega t$ are lin. ind. fcns,
identity can hold iff

$$* \quad \begin{cases} A = C\cos\alpha \\ B = C\sin\alpha \end{cases}$$

iff $A^2+B^2 = C^2 \quad \left(= C^2(\cos^2\alpha + \sin^2\alpha)\right)$

and α is the
angle in the $\mathbb{R}^2$ triangle



since $*$ holds iff

$C = \sqrt{A^2+B^2}$

$A/C = \cos\alpha$

$B/C = \sin\alpha$

(the hypotenuse vector C
can point into any
quadrant, depending
on the signs of A and B).

$n = 29$

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Exam 1 scores