Math 2280-1
Wed 5 April

§7.1-7.3 Laplace transform

Fill in the table entries for
$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

(1a) differentiating $f(t)$ k times corresponds to multiplying $F(s)$ by s^k, roughly speaking, even for k negative!

remains to show that for
$$g(t) = \int_0^t f(\tau)\,d\tau\,, \quad G(s) = \frac{F(s)}{s}$$

but $g'(t) = f(t)$ and $g(0)=0$

so $\underbrace{\mathcal{L}\{g'(t)\}}_{F(s)} = s\,G(s) - 0 = s\,G(s)$, i.e. $G(s) = \frac{F(s)}{s}$ ■

(1b) Conversely, taking derivatives of $F(s)$ corresponds to multiplying $f(t)$ by powers of t (and a ±)!

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\int_0^\infty e^{-(s+\Delta s)t} f(t)\,dt - \int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \lim_{\Delta s\to 0} \int_0^\infty f(t)\left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right] dt$$

$$\downarrow \quad \frac{d}{ds} e^{-st} = -s e^{-st}$$

$$\stackrel{!}{=} \int_0^\infty f(t)(-s)e^{-st}\,dt = -s\,F(s) \quad ■$$

so $\mathcal{L}\{t^2 f(t)\}(s) = -\left[\mathcal{L}\{t f(t)\}(s)\right]'$
$= -(-F'(s))'$
$= F''(s)$

etc.

(get $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)$ formula similar to (1a) reasoning)

HW for Fri 4/14

§7.2 19, (20) (28) 31
§7.3 (3) 7 (8) (17, 20, 31)
§7.4 (2) (3) 9 (10) 15 (16) (29) (36)
§7.5 (11, 31) } we'll see!
§7.6 (1, 8) }

	$f(t)$	$F(s)$	
	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	$\mathcal{L}$ is linear!
	1	$1/s$	
	e^{at}	$1/(s-a)$	$\mathrm{Re}\, s > \mathrm{Re}\, a$
(1a)	$f'(t)$	$sF(s) - f(0)$	
	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
	$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
		etc	
	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	
(1b)	$t f(t)$	$-F'(s)$	
	$t^2 f(t)$	$F''(s)$	
	$t^3 f(t)$	$-F'''(s)$	
	$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
		etc	
(2a)	$u(t-a)$	e^{-as}/s	
	$u(t-a)f(t-a)$	$e^{-as}F(s)$	
(2b)	$e^{at} f(t)$	$F(s-a)$	
	1	$1/s$	
	t	$1/s^2$	
	t^2	$2/s^3$	
	t^n	$n!/s^{n+1}$	
	$\cos kt$	$s/(s^2+k^2)$	
	$\sin kt$	$k/(s^2+k^2)$	
	$\cosh kt$	$s/(s^2-k^2)$	
	$\sinh kt$	$k/(s^2-k^2)$	
	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	
	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	
	$t^n e^{at}$	$n!/(s-a)^{n+1}$	
	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$	
	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$	

2b $\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$ ■

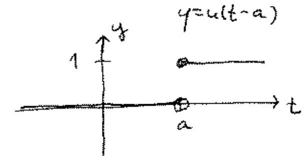
application: $\mathcal{L}\{\cos kt\}(s) = \dfrac{s}{s^2+k^2}$, $\mathcal{L}\{\sin kt\}(s) = \dfrac{k}{s^2+k^2}$

so $\mathcal{L}\{e^{at}\cos kt\}(s) = \dfrac{s-a}{(s-a)^2+k^2}$

and $\mathcal{L}\{e^{at}\sin kt\}(s) = \dfrac{k}{(s-a)^2+k^2}$

So, what is $\mathcal{L}^{-1}\left(\dfrac{4}{s^2+2s+5}\right)$? Complete the square!

graph of $y=u(t-a)$



2a The unit step function $u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

and its translation $u(t-a) = \begin{cases} 1 & t-a \geq 0 \text{ i.e. } t \geq a \\ 0 & t-a < 0 \text{ i.e. } t < a \end{cases}$

are used to turn forcing terms on and off in applied DE's

$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty u(t-a)e^{-st}\,dt = \int_a^\infty e^{-st}\,dt \quad \left(\text{since } \int_0^a = 0!\right)$$

$$= \left.\frac{e^{-st}}{-s}\right]_{t=a}^{\infty} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = \int_0^\infty \text{stuff}\,dt = \int_a^\infty f(t-a)e^{-st}\,dt \underset{\substack{\tau = t-a \\ d\tau = dt \\ (t = \tau + a)}}{=} \int_0^\infty f(\tau)\underbrace{e^{-s(\tau+a)}}_{e^{-s\tau}e^{-as}}\,dt$$

$$= e^{-as}F(s) \quad ■$$

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application of 1a or 1b:

$\mathcal{L}\{1\}(s) = \frac{1}{s}$

so $\mathcal{L}\{t\cdot 1\}(s) = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$

so $\mathcal{L}\{t\cdot t\}(s) = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2!}{s^3}$

by induction, $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$

So, by 2b, $\mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

example (§7.3 p.459) unforced, damped spring:

$$\begin{cases} x'' + 6x' + 34x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$$

$\mathcal{L}$: $s^2 X(s) - 3s - 1 + 6(sX(s) - 3) + 34X(s) = 0$

$$X(s)(s^2 + 6s + 34) = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$

$$= \frac{3(s+3) + 10}{(s+3)^2 + 25}$$ ← complete the linear (?) ← complete the square

$$= 3\frac{(s+3)}{(s+3)^2+25} + 2 \cdot \frac{5}{(s+3)^2+25}$$

$$\mathcal{L}\{\cos 5t\}(s) = \frac{s}{s^2+5^2}$$

so $\mathcal{L}\{e^{-3t}\cos 5t\}(s) = \frac{s+3}{(s+3)^2+25}$

So $\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$

example (§7.3 p.460)

$$\begin{cases} y'' + 4y' + 4y = t^2 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

so $y(t) = 3/8 - \frac{1}{2}t + \frac{1}{4}t^2 - \frac{3}{8}e^{-2t} - \frac{1}{4}te^{-2t}$!!

$\mathcal{L}$: $s^2 Y(s) + 4sY(s) + 4Y(s) = \frac{2}{s^3}$

$$Y(s)(s+2)^2 = \frac{2}{s^3}$$

$$Y(s) = \frac{2}{s^3(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2} + \frac{E}{(s+2)^2} = \frac{As^2(s+2)^2 + Bs(s+2)^2 + \cdots}{s^3(s+2)^2}$$

$$2 = As^2(s+2)^2 + Bs(s+2)^2 + C(s+2)^2 + Ds^3(s+2) + Es^3$$

$s = -2 \Rightarrow 2 = -8E$; $E = -1/4$

$s = 0 \Rightarrow 2 = 4C$ $C = 1/2$

now equate coeffs: (using C, E values)

$2 = 1(4(\frac{1}{2}))$ ✓

$+ s(4B + 2) \Rightarrow B = -1/2$

$+ s^2(4A + 4B + \frac{1}{2}) \Rightarrow 4A - 3/2 = 0 \Rightarrow A = 3/8$

$+ s^3(4A + B + 2D - \frac{1}{4})$

$+ s^4(A + D) \Rightarrow D = -3/8$