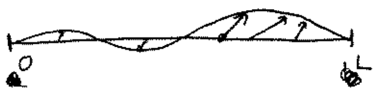
Math 2280-1

Mon 4/24

§9.6

Vibrating strings & the wave equation



vibrating string.

We assume tension in string changes as it is stretched,

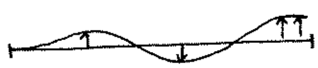
i.e. $T = \mathcal{T}(\rho)$ $\quad \rho$ = density $\quad$ (probably $\mathcal{T}(\rho) \searrow$ fcn ρ)

$$T = \underset{T_0}{\underbrace{\mathcal{T}(\rho_0)}} + \mathcal{T}'(\rho_0)(\rho - \rho_0)$$

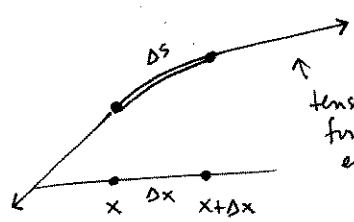
linearized model; T_0, ρ_0 for stationary (but stretched) string.

<u>case 1</u> transverse (vertical) vibrations

$y = y(x,t)$ = vertical displacement



isolate a tiny piece, apply Newton's Law, linearize, to deduce wave eqtn:



tension force on ends

$$\underbrace{\rho \Delta s}_{\text{mass}} \; \underbrace{y_{tt}}_{\text{accel}} = \overbrace{T \frac{y_x}{\sqrt{1+y_x^2}} \Bigg]_x^{x+\Delta x}}^{\text{net forces}}$$

tension

vertical component of unit tang. vector to profile curve

$$\begin{bmatrix} 1 \\ y_x \end{bmatrix} \frac{1}{\sqrt{1+y_x^2}}$$

linearize

$$\frac{\Delta s}{\Delta x} \approx \sqrt{1+y_x^2} \approx 1$$

(y_x small $\Rightarrow y_x^2$ negligable)

$$\rho_0 (\Delta x)\, y_{tt} \approx T_0 \left(y_x(x+\Delta x) - y_x(x) \right)$$

$$\rho \Delta s = \rho_0 \Delta x$$

$$\Rightarrow \rho = \rho_0 \frac{ds}{dx} = \rho_0 \sqrt{1+y_x^2} \approx \rho_0$$

$$\Rightarrow T \approx T_0$$

$$\rho_0 y_{tt} \approx T_0 \left(\frac{y_x(x+\Delta x) - y_x(x)}{\Delta x} \right)$$

$$y_{tt} = \left(\frac{T_0}{\rho_0} \right) y_{xx}$$

$$\boxed{y_{tt} = a^2 y_{xx}} \qquad a = \sqrt{\frac{T_0}{\rho_0}}$$

What does "a" mean?

Special solution:

Let $f(z)$ a function of 1-variable.

consider $y(x,t) = f(x-at)$

$$y_t = f'(x-at)(-a) \qquad \text{chain rule!}$$

$$y_{tt} = f''(x-at)\, a^2$$

$$y_x = f'(x-at)\, 1$$

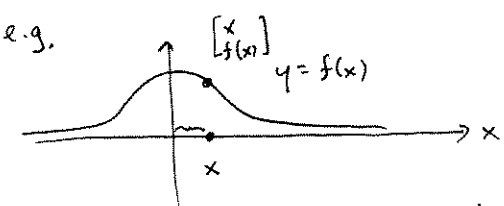
$$y_{xx} = f''(x-at)\, 1$$

$$y_{tt} = a^2 y_{xx} \quad !$$

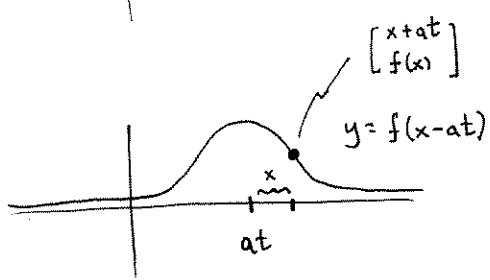
(also, $y(x,t) = f(x+at)$ solves the wave equation.)

$y(x,t) = f(x-at)$ is a wave (with constant "profile") moving to the right $\boxed{\text{with speed } a!}$

e.g.



profile at $t=0$



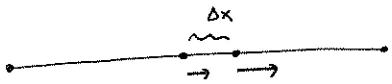
profile at time t

$y(x,t) = f(x+at)$ is a wave moving to the left!

(It will turn out that every solution to the wave eqtn $y_{tt} - a^2 y_{xx} = 0$ is a superposition of waves traveling to the left or right with speed a!)

In case 1, $a = \sqrt{\frac{T_0}{\rho_0}}$, so higher tension $\Rightarrow$ faster speed
lower density $\Rightarrow$ faster speed.

Case 2 parallel (horizontal) vibrations



$u(x,t)$ = horizontal displacement.

$$\Delta s = (x+\Delta x) + u(x+\Delta x, t) \quad - \left[x + u(x,t) \right]$$

$$\Delta s = \Delta x + \Delta u$$

$$\frac{ds}{dx} = 1 + u_x$$

$$\rho \, ds = \rho_0 \, dx$$

$$\rho = \rho_0 \frac{dx}{ds} = \rho_0 \frac{1}{1+u_x} \approx \rho_0 (1 - u_x) \qquad \text{(linearize!)}$$

so $T = T(\rho) = T(\rho_0) + T'(\rho_0)(-\rho_0 u_x)$

Newton:

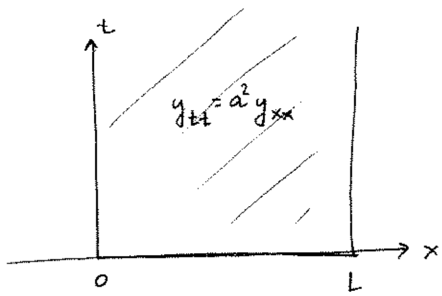
$$\underbrace{(\rho \Delta s)}_{= \cancel{\rho_0} \Delta x} u_{tt} = T \Big]_x^{x+\Delta x} = T(\rho_0) + T'(\rho_0)\left(-\rho_0 u_x(x+\Delta x, t)\right) - \left[T(\rho_0) + T'(\rho_0)(-\rho_0 u_x(x,t))\right]$$

$$\doteq -\cancel{\rho_0} T'(\rho_0) \left[u_x(x+\Delta x, t) - u_x(x,t) \right]$$

$$\boxed{u_{tt} = \left[-T'(\rho_0)\right] u_{xx}}$$

also a wave eqtn,
but with diffrent speed $a = \sqrt{-T'(\rho_0)}$

Natural IBVP's for wave eqtn:



$$\begin{cases} y_{tt} = a^2 y_{xx} & t > 0,\ 0 < x < L \\ y(x,0) = f(x) & \text{initial displacement} \\ y_t(x,0) = g(x) & \text{initial velocity} \end{cases}$$

plus ① $y(0,t) = y(L,t) = 0$ fixed endpoints
② $y_x(0,t) = y_x(L,t) = 0$ free endpoints

Separated soltns:

$$y(x,t) = X(x)T(t)$$

$$XT'' = a^2 X'' T$$

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

interesting case $\lambda = -r^2$ yields $X(x) = \text{span}\{\cos rx, \sin rx\}$
$T(t) = \text{span}\{\cos(art), \sin(art)\}$.

Use Fourier series & superposition: $(2\times 2 = 4)$

① $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0) = f(x)$ $(y_t(x,0) = 0)$
$y(0,t) = y(L,t) \equiv 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0) = g(x)$ $(y(x,0) = 0)$
$y(0,t) = y(L,t) = 0$

② $\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0) = f(x)$ $(y_t(x,0) = 0)$
$y_x(0,t) = y_x(L,t) = 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0) = g(x)$ $(y(x,0) = 0)$
$y_x(0,t) = y_x(L,t) = 0$

Slinky math

assume equil. slinky has length ≈ 0, and is Hooke's-like and mass m.
So if it's stretched to length L

$$\rho = \frac{m}{L}$$

$$T = kL \qquad k = \text{Hooke's const.}$$

transverse oscillations

$$a = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{kL}{m/L}} = L\sqrt{\frac{k}{m}}$$

parallel oscillations:

$$T = f(\rho) = kL = \frac{km}{\rho}$$

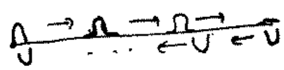
$$T'(\rho) = -\frac{km}{\rho^2}$$

!!

$$a = \sqrt{-T'(\rho)} = \sqrt{\frac{km}{\rho^2}} = \sqrt{\frac{km}{m^2/L^2}} = L\sqrt{\frac{k}{m}}$$

1 test this

pulse period:



← fixed endpoint
waves return upside down.
Why?

time for a wave to go back and forth (i.e. distance $2L$) is

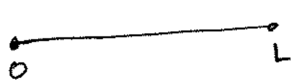
$$(\text{speed})(\text{time}) = 2L$$

$$\text{time} = \frac{2L}{L\sqrt{k/m}} = 2\sqrt{\frac{m}{k}} \quad \text{is independent of } L \text{ !!}$$

2 test this

alternate way to measure speed:

fundamental mode (fixed endpoint)



$$y(x,t) = \sin\left(\frac{x\pi}{L}\right)\cos\left(\frac{a\pi}{L}t\right) \qquad \text{same } a!$$

$$\left(y = \frac{1}{2}\left[\sin\left(\frac{\pi}{L}(x-at)\right) + \sin\left(\frac{\pi}{L}(x+at)\right)\right] \right)$$

$$\text{one period} = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{a\pi}{L}\right)}$$

$$= \frac{2L}{a} = \frac{2L}{L\sqrt{k/m}} = 2\sqrt{\frac{m}{k}}$$

$$= \text{pulse period}$$

3 test this