Math 2280-1
Tues 4/18

Resonance mysteries revisited.
We shall study

$$x''(t) + 16\,x(t) = f(t)$$

If $x_0 = v_0 = 0$

$$X(s) = \frac{1}{s^2+16}\,F(s)$$

$$x(t) = \frac{1}{4}\int_0^t \sin(4(t-\tau))\,f(\tau)\,d\tau$$

convolution solution

$\omega_0 = 4$

$T_0 = \frac{2\pi}{\omega_0} = \frac{\pi}{2}$ natural period.

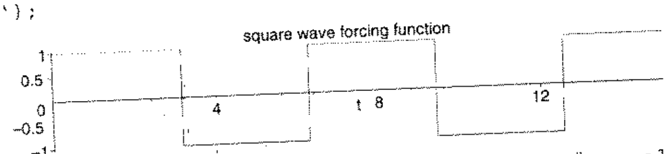
We shall force with

$f(t)$ = square wave with period 2π

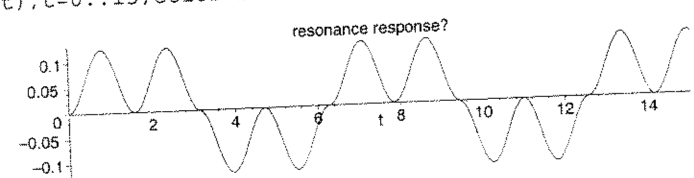
$g(t)$ = 2π-periodic sawtooth fcn.

Here are the results:

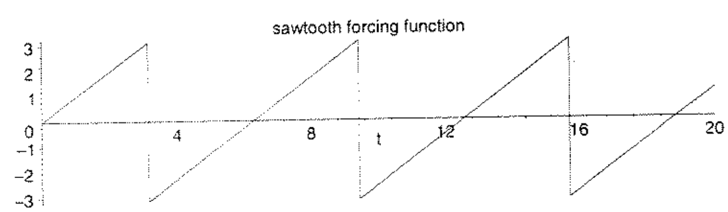
```
> plot(f(t),t=0..15,color=black,title='square wave forcing
  function');
```



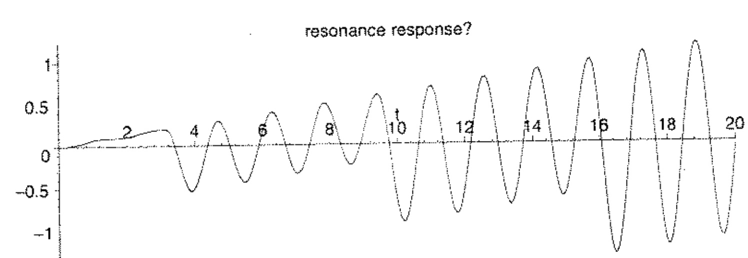

```
> x:=t->.25*int(sin(4*(t-tau))*f(tau),tau=0..t):    #convolution
  formula for the solution
> plot(x(t),t=0..15,color=black,title='resonance response?');
```




```
> g:=t->t-2*Pi*sum(Heaviside(t-(2*n+1)*Pi),
  n=0..3):
> plot(g(t),t=0..20,color=black, title='sawtooth forcing function');
```




```
> y:=t->.25*int(sin(4*(t-tau))*g(tau),tau=0..t):
> plot(y(t),t=0..20,color=black, title='resonance response?');
```



Explain these results using Fourier series!

① Square wave

$$f \sim \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots \right)$$

```
> f:=t->-1+2*Heaviside(t); #square wave on [-Pi,Pi]
  a0:=(1/Pi)*int(f(t),t=-Pi..Pi);
```

$$f := t \to -1 + 2\,\mathrm{Heaviside}(t)$$
$$a0 := 0$$

```
> with(linalg):
> a:=vector(10);b:=vector(10);
```

$$a := \mathrm{array}(1\,..\,10, [\,])$$
$$b := \mathrm{array}(1\,..\,10, [\,])$$

```
>
  for k from 1 to 10 do
  a[k]:=(1/Pi)*int(f(t)*cos(k*t),t=-Pi..Pi):
  b[k]:=(1/Pi)*int(f(t)*sin(k*t),t=-Pi..Pi):
  od:
> evalm(a);evalm(b);
```

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$
$$\left[\frac{4}{\pi}, 0, \frac{4}{3\pi}, 0, \frac{4}{5\pi}, 0, \frac{4}{7\pi}, 0, \frac{4}{9\pi}, 0\right]$$

Auxillary computation:

$x'' + 16x = 0 \quad : \quad x_H = A\cos 4t + B\sin 4t$

$x'' + 16x = \sin kt \qquad X(s) = \frac{k}{s^2+k^2}\,\frac{1}{s^2+16}$

$k \neq 4 \qquad x_p(t) = \frac{1}{16-k^2}\sin kt$

$k = 4 \qquad x_p(t) = -\frac{1}{8}t\cos 4t$

if $k \neq 4$,
$X(s) = \frac{k}{k^2-16}\left[\frac{1}{s^2+16} - \frac{1}{s^2+k^2}\right]$

if $k = 4$
$X(s) = \frac{4}{(s^2+16)^2}$

Now find a particular sol'n to

$x'' + 16x = f(t)$

(2) sawtooth

$$g \sim 2\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \dots\right)$$

Now for g(t):

```
> g:=t->t;  #definition on [-Pi,Pi]
  a0:=(1/Pi)*int(f(t),t=-Pi..Pi);
```

$$g := t \to t$$

$$a0 := 0$$

```
> for k from 1 to 10 do
  a[k]:=(1/Pi)*int(g(t)*cos(k*t),t=-Pi..Pi):
  b[k]:=(1/Pi)*int(g(t)*sin(k*t),t=-Pi..Pi):
  od:evalm(a);evalm(b);
```

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

$$\left[2, -1, \frac{2}{3}, \frac{-1}{2}, \frac{2}{5}, \frac{-1}{3}, \frac{2}{7}, \frac{-1}{4}, \frac{2}{9}, \frac{-1}{5}\right]$$

Which is consistent with the Fourier expansion

$$g \sim 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} \sin(n\,t)}{n}\right)$$

So, find a particular sol'n to

$$x'' + 16x = g(t)$$

Formulate a theorem for the forced harmonic oscillator

$$x'' + \omega_0^2 x = f(t)$$

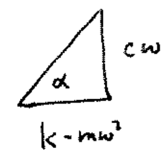
where f is periodic, with period T. Exactly when do you get resonance?

You can also apply these ideas with damping: (p 604-605)

$$mx'' + cx' + kx = F(t)$$

special case §3.6, if $F(t) = F_0 \sin \omega t$

then $$x_p(t) = x_{sp}(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \alpha)$$



So, if $$F \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} t\right)$$

Find $x_{sp}(t)$