Math 2280-1
Monday 4/17

9.3, mostly.
• work out Fourier series for 2L-periodic fns, and examples, on Friday notes

• Fourier sine and cosine series:
  \[ f \colon [0, L] \to \mathbb{R} \]
  • you can extend \( f \colon [-L, L] \to \mathbb{R} \) as an even fn; \( f(-t) = f(t) \)
    and then to \( \mathbb{R} \) as a 2L-periodic fn
    \[ \Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi}{L} t \right) \]
    cosine series for \( f \)
  • you can extend \( f \colon [-L, L] \to \mathbb{R} \) as an odd fn; \( f(-t) = -f(t) \)
    and then to \( \mathbb{R} \) as a 2L-periodic fn
    \[ \Rightarrow f \sim \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{L} t \right) \]
    sine series for \( f \)

examples (we can sketch)

1. let \( f(t) = 1, 0 \leq t \leq 1 \)
   What is the cosine series for \( f \)?

2. let \( g(t) = t, 0 \leq t \leq 1 \)
   What is the cosine series for \( g \)?

   What is the sine series for \( g \)?
Theorems about differentiating and integrating Fourier series term by term:

**Theorem 1:** Let \( f(t) \) be continuous and 2\( L \)-periodic, with

Fourier series
\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi t}{L} \right) + b_n \sin \left( \frac{n \pi t}{L} \right)
\]

If \( f''(t) \) is piecewise smooth, then \( f' \) has Fourier series
\[
f'(t) \sim \sum_{n=1}^{\infty} -a_n \left( \frac{n \pi}{L} \right) \sin \left( \frac{n \pi t}{L} \right) + b_n \left( \frac{n \pi}{L} \right) \cos \left( \frac{n \pi t}{L} \right)
\]

(i.e. obtained from \( f' \)'s by termwise differentiation)

(The issue here is that we're trying to pass \( \frac{d}{dt} \) through an infinite sum)

**Proof:** \( f' \) is 2\( L \)-periodic and p.w. smooth, so has a Fourier series
\[
f'(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n \pi t}{L} \right) + B_n \sin \left( \frac{n \pi t}{L} \right)
\]

\[
A_0 = \frac{1}{L} \int_{-L}^{L} f'(t) \, dt = \frac{1}{L} f(t) \bigg|_{-L}^{L} = 0 \quad \text{since} \quad f \text{ is 2L-periodic}
\]

\[
A_n = \frac{1}{L} \int_{-L}^{L} f'(t) \cos \left( \frac{n \pi t}{L} \right) \, dt = \frac{1}{L} \int_{-L}^{L} \left[ f(t) \cos \left( \frac{n \pi t}{L} \right) \right] \, dt - \frac{1}{L} \int_{-L}^{L} f(t) \frac{n \pi}{L} \sin \left( \frac{n \pi t}{L} \right) \, dt
\]

\[
B_n \text{ computation analogous}
\]

**Theorem 2:** If \( f \) is p.w. cont., 2\( L \)-periodic, with Fourier series
\[
f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi t}{L} \right) + b_n \sin \left( \frac{n \pi t}{L} \right)
\]

then the antiderivative
\[
\int_{0}^{t} f(s) \, ds \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left( \frac{L}{n \pi} \right) \sin \left( \frac{n \pi t}{L} \right) + b_n \left( \frac{L}{n \pi} \right) \left[ \cos \left( \frac{n \pi t}{L} \right) + C \right]
\]

\( C \) is the constant of integration.

**Proof:** (page 543.)
Example: In HW, you could use these facts to derive the Fourier series for

\[ f(t) = t^2, \quad -1 \leq t \leq 1 \]

if extended to be 2-periodic.