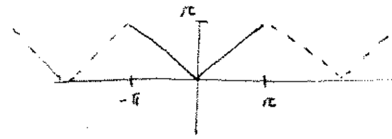
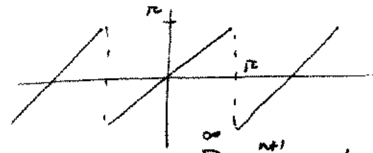
Math 2280-1
Monday 4/17
§9.3, mostly.

- work out Fourier series for 2L-periodic fcns, and examples, on Friday notes



$$f \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos nt$$



$$f \sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$$

- Fourier sine and cosine series:

If $f:[0,L]\to\mathbb{R}$

- you can extend $f:[-L,L]\to\mathbb{R}$ as an <u>even fcn</u>: $f(-t):=f(t)$ and then to $\mathbb{R}$ as a 2L-periodic fcn

$$\Rightarrow \quad f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}t\right) \qquad \text{\underline{cosine series} for } f$$

- you can extend $f:[-L,L]\to\mathbb{R}$ as an <u>odd</u> fcn: $f(-t):=-f(t)$, and then to $\mathbb{R}$ as a 2L-periodic fcn

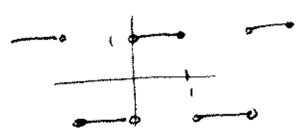
$$\Rightarrow f \sim \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}t\right) \qquad \text{\underline{sine series} for } f$$

examples (we can steal)

① Let $f(t)=1$, $0\le t\le 1$

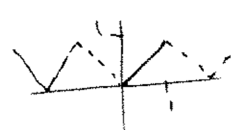
What is the cosine series for f?

What is the sine series for f?

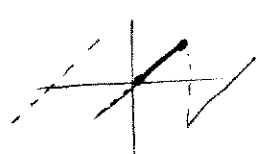


② Let $g(t)=t$, $0\le t\le 1$

What is the cosine series for g?



What is the sine series for g?



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Theorems about differentiating and integrating Fourier series term by term:

<u>Theorem 1</u> : Let $f(t)$ be continuous an $2L$-periodic, with Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

<u>If</u> $f'(t)$ is piecewise smooth, then f' has Fourier series

$$f' \sim \sum_{n=1}^{\infty} -a_n\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}t\right) + b_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}t\right)$$

(i.e. obtained from f's by termwise differentiation)
(The issue here is that we're trying to pass $\frac{d}{dt}$ through an <u>infinite</u> sum)

<u>proof</u> : f' <u>is</u> $2L$-periodic and p.w. smooth, so has a Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}t\right) + B_n \sin\left(\frac{n\pi}{L}\right)t$$

$$A_0 = \frac{1}{L}\int_{-L}^{L} f'(t)\,dt = \frac{1}{L}f(t)\Big]_{-L}^{L} = 0 \quad \text{since } f \text{ is } 2L\text{-periodic.}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f'(t)\underbrace{\cos\left(\frac{n\pi}{L}t\right)}_{u}dt \;\underset{\text{int by parts!}}{=}\; \underbrace{\frac{1}{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\Big]_{-L}^{L}}_{0 \text{ by periodicity}} - \underbrace{\frac{1}{L}\int_{-L}^{L} f(t)\left(\frac{n\pi}{L}\right)\left(-\sin\left(\frac{n\pi}{L}t\right)\right)dt}_{- - \frac{n\pi}{L} b_n \ \checkmark}$$

B_n computation analogous. ∎

<u>Theorem 2</u> : If f is p.w. cont., $2L$-periodic, with Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

then the antiderivative

$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi}{L}t\right) + b_n\left(\frac{L}{n\pi}\right)\left[-\cos\left(\frac{n\pi}{L}t\right) + 1\right]$$

integrated term by term.

series on the right converges to this antideriv!

(proof : page 599.)

<u>Example</u>: In HW, you could use these thms to derive the Fourier series for

$f(t) = t^2, \quad -1 \le t \le 1$

f extended to be 2-periodic