Math 2280-1
Fri 4/14

§9.1-9.2 Fourier series

HW for 4/21

9.1 3, 6, 7, 10, 12, 13, 15, 17, 20, 27, 28, 29, 30
9.2 2, 9
9.3 1, 9, 17, 19, 20
9.4 1, 7, 9, 13

(there will be a final HW assignment covering 9.5 & 9.6)

Set up:

$f: \mathbb{R} \to \mathbb{R}$ 2π periodic

Fourier coeff's:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

1 Notice: we motivated Fourier series using 2270 orthogonality ideas
our text makes the linear alg connection less explicitly, but it's still there:

If $\quad f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$

then, says text, you can recover a_0, a_n, b_n as follows
(assuming $\int\sum = \sum\int$)

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dt + \sum_1 a_n \cancel{\int_{-\pi}^{\pi}\cos nt}^{\,0} + \sum b_n \cancel{\int_{-\pi}^{\pi}\sin nt\,dt}^{\,0}$$

$$= \frac{a_0}{2}\cdot 2\pi = \pi a_0 \Rightarrow \text{formula for } a_0.$$

$$\int_{-\pi}^{\pi} f(t)\cos kt\,dt = 0 + \sum_n a_n \underbrace{\int_{-\pi}^{\pi}\cos nt\cos kt\,dt}_{=0 \text{ if } n\neq k.} + \sum_n b_n \cancel{\int_{-\pi}^{\pi}\sin nt\cos kt\,dt}^{\,0}$$

if $n=k$ get $\frac{1}{2}\cdot 2\pi$

$$\Rightarrow \int_{-\pi}^{\pi} f(t)\cos kt\,dt = a_k\pi \Rightarrow \text{formula for } a_k$$

(formula for b_k analogous)

Finish Wed. notes

- convergence theorems
- magic identities from examples 1, 2

2 Notice, we assumed f was 2π-periodic.

What if $g(u)$ is $2L$-periodic?

Then define

$$f(t) = g\left(\frac{tL}{\pi}\right)$$

as t varies by 2π, $\boxed{u = \frac{tL}{\pi}}$ varies by $2L$ $\qquad \boxed{t = \frac{\pi u}{L}}$

Then f is 2π-periodic

so $\displaystyle f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$ as on page 1.

so $\displaystyle g(u) = f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}u\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}u\right)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g\left(\frac{tL}{\pi}\right)dt \underset{\substack{u = \frac{tL}{\pi} \\ du = \frac{L}{\pi}dt}}{=} \frac{1}{\pi}\frac{\pi}{L}\int_{-L}^{L} g(u)\,du = \frac{1}{L}\int_{-L}^{L} g(u)\,du$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g\left(\frac{tL}{\pi}\right)\cos(nt)\,dt = \frac{1}{L}\int_{-L}^{L} g(u)\cos\left(n\frac{\pi}{L}u\right)du$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(u)\cos\left(\frac{n\pi}{L}u\right)du$$

Summary:

$f: \mathbb{R} \to \mathbb{R}$ $\quad 2L$-periodic

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

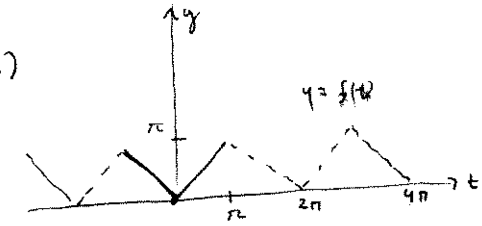
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right)$$

and same convergence theorems hold (page 5 Wed 4/12)

Example 3: From page 3 Wed: (Example 1)

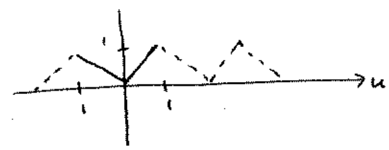


$f(t) = |t| \quad -\pi \le t \le \pi$

f extended as 2π-periodic

$$f \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n \text{ odd} \\ n=1,3,\dots}} \frac{\cos nt}{n^2}$$

Find the Fourier series for the rescaled tent fcn



$g(u) = |u| \quad -1 \le u \le 1$

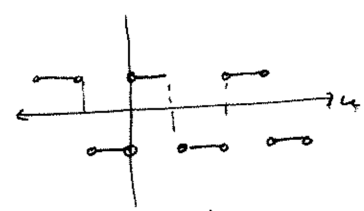
g extended as $2L = 2$ periodic

$$u = \frac{t}{\pi}$$

$$g(u) = \frac{1}{\pi} f(t) \sim \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

$$\boxed{g(u) \sim \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{\cos(n\pi u)}{n^2}}$$

Example 4 $\quad h(u) := g'(u) = \begin{cases} -1 & -1 < u < 0 \\ +1 & 0 < u < 1 \end{cases}$ is 2-periodic



"square wave".

If we differentiate the Fourier series for $g(u)$ term by term, do we get the Fourier series for h?

i.e. is $\quad h(u) \sim -\frac{4}{\pi^2} \sum_{n \text{ odd}} -\sin(n\pi u) \frac{n\pi}{n^2}$

$$= \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(n\pi u)}{n} \quad ?$$

$\left(b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \right)$

Work this out on the next page!

- Why are all a_n $(n=0,1,\dots)$ zero?
- $$b_n = \frac{1}{L}\int_{-L}^{L} \sin(n\pi t)\, f(t)\, dt$$

$$=$$