Math 2280-1
Tues April 11

Cancel §7.6 HW problems

§7.4-7.5

convolution (for Laplace transf.)

$$f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau$$
$$= g*f(t)$$

Theorem

$$\mathcal{L}\{f*g(t)\}(s) = F(s)G(s)$$

example: Verify the theorem
for $f(t) = \sin t$
$g(t) = \cos t$

[you may need the trig identity
$\sin^2\tau = \frac{1-\cos 2\tau}{2}$]

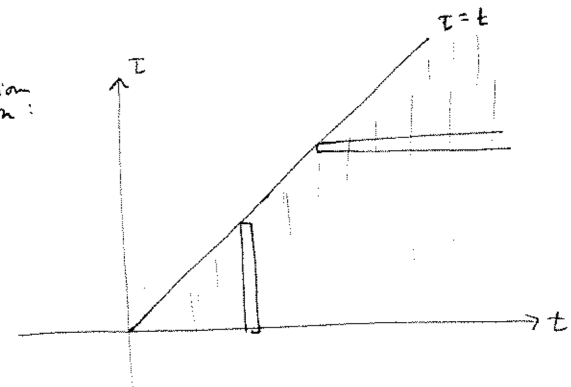
	$f(t)$	$F(s)$	
	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	$\mathcal{L}$ is linear!
	1	$1/s$	
	e^{at}	$1/(s-a)$	$\operatorname{Re} s > \operatorname{Re} a$
1a	$f'(t)$	$sF(s) - f(0)$	
	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
	$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
		etc	
	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	
1b	$t f(t)$	$-F'(s)$	
	$t^2 f(t)$	$F''(s)$	
	$t^3 f(t)$	$-F'''(s)$	
	$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
		etc	
2a	$u(t-a)$	e^{-as}/s	today?
	$u(t-a)f(t-a)$	$e^{-as}F(s)$	
2b	$e^{at}f(t)$	$F(s-a)$	
	1	$1/s$	
	t	$1/s^2$	
	t^2	$2/s^3$	
	t^n	$n!/s^{n+1}$	
	$\cos kt$	$s/(s^2+k^2)$	
	$\sin kt$	$k/(s^2+k^2)$	
	$\cosh kt$	$s/(s^2-k^2)$	
	$\sinh kt$	$k/(s^2-k^2)$	
	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	
	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	
	$t^n e^{at}$	$n!/(s-a)^{n+1}$	
	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$	
	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$	
	$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$	
	$(f*g)(t)$	$F(s)G(s)$	today!

proof of convolution theorem:
(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right] d\tau$$

$\tilde{t} = t - \tau$
$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$

Application: for any nonhomogeneous, const. coeff linear DE or system, there is a convolution formula for the solution (must be equivalent to variation of parameters ?!)

example

$$\begin{cases} mx'' + cx' + kx = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$\mathcal{L}$: $X(s)\left[ms^2 + cs + k\right] = F(s) + (sx_0 + v_0)m + x_0 c$

$$X(s) = \frac{F(s)}{ms^2+cs+k} + \frac{(sx_0+v_0)m + x_0 c}{ms^2+cs+k}$$

↑ use convolution formula to invert

↑ table

subexample

$$\begin{cases} x'' + x = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$\mathcal{L}$: $X(s)(s^2+1) = F(s)$

$$X(s) = \frac{1}{s^2+1} F(s)$$

$$x(t) = (\sin * f)(t) = \int_0^t f(\tau)\sin(t-\tau)\,d\tau = \int_0^t (\sin\tau) f(t-\tau)\,d\tau$$

see handout & play the resonance game.

Math 2280-1
April 11, 2006
Guess the resonance game, using convolution formula, section 7.4

```
> with(plots):with(inttrans):
    #the library inttrans includes Laplace
```

We are considering the undamped forced harmonic oscillator

$$x''(t) + x(t) = f(t)$$

with initial data x(0)=v(0)=0. When we take the Laplace transform of this equation we deduce

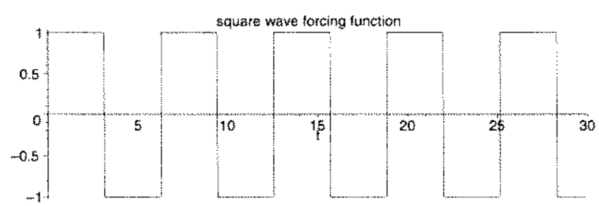
$$X(s) = \frac{F(s)}{s^2+1}$$

so that the convolution theorem implies $x(t) = \sin * f(t)$. Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of 2π. Here's a square wave forcing function:

```
> f:=t->-1+2*sum((-1)^n*Heaviside(t-n*Pi),n=0..10);
   #Heaviside was an early user of the unit step function
   #and so Maple names it after him
```

$$f := t \to -1 + 2\left(\sum_{n=0}^{10} (-1)^n \text{Heaviside}(t - n\pi)\right)$$

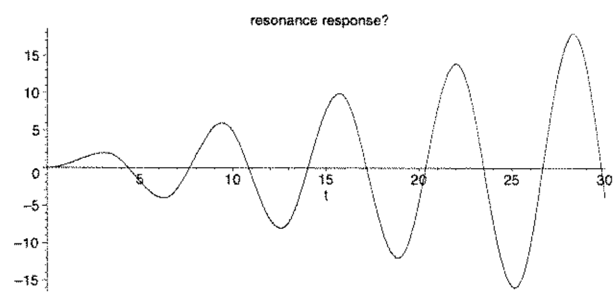
```
> plot(f(t),t=0..30,color=black,title='square wave forcing
  function');
```




```
> x:=t->int(sin(t-tau)*f(tau),tau=0..t);
     #convolution formula for the solution
```

$$x := t \to \int_0^t \sin(t-\tau)\,f(\tau)\,d\tau$$

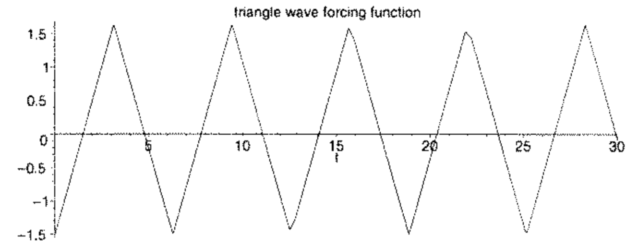
```
> plot(x(t),t=0..30,color=black,title='resonance response?');
```




```
> g:=t->int(f(u),u=0..t)-1.5;
  #this should be a triangle wave...
```

$$g := t \to \int_0^t f(u)\,du - 1.5$$

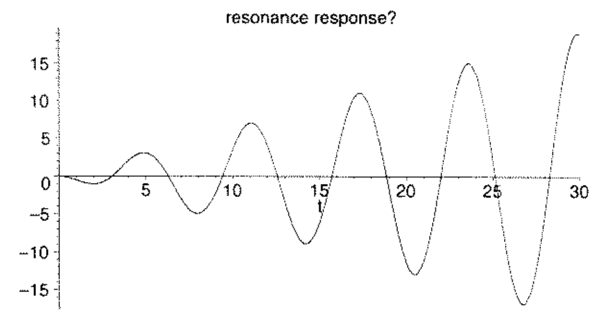
```
> plot(g(t),t=0..30,color=black, title='triangle wave forcing
  function');
```




```
> y:=t->int(sin(t-tau)*g(tau),tau=0..t);
```

$$y := t \to \int_0^t \sin(t-\tau)\,g(\tau)\,d\tau$$

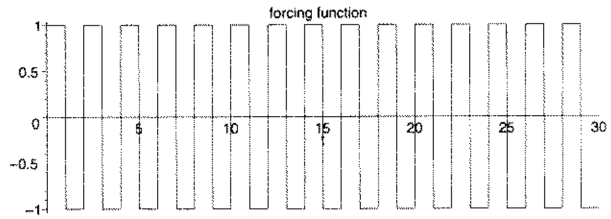
```
> plot(y(t),t=0..30,color=black, title='resonance response?');
```



```
> h:=t->-1+2*sum((-1)^n*Heaviside(t-n),n=0..30);
```

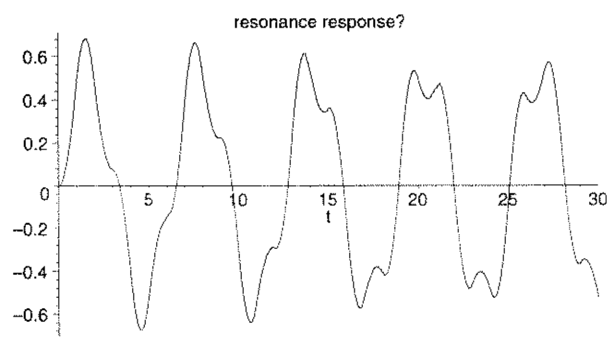
$$h := t \to -1 + 2\left(\sum_{n=0}^{30} (-1)^n \,\mathrm{Heaviside}(t-n)\right)$$

```
> plot(h(t),t=0..30,color=black, title='forcing function');
```




```
> z:=t->int(sin(t-tau)*h(tau),tau=0..t);
  plot(z(t),t=0..30,color=black,title='resonance response?');
```

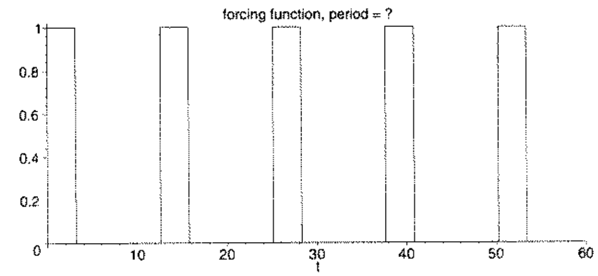
$$z := t \to \int_0^t \sin(t-\tau)\,\mathrm{h}(\tau)\,d\tau$$




```
> k:=t->sum(Heaviside(t-4*Pi*n)-Heaviside(t-4*Pi*n-Pi),
  n=0..5);
```

$$k := t \to \sum_{n=0}^{5} (\mathrm{Heaviside}(t - 4\,n\,\pi) - \mathrm{Heaviside}(t - 4\,n\,\pi - \pi))$$

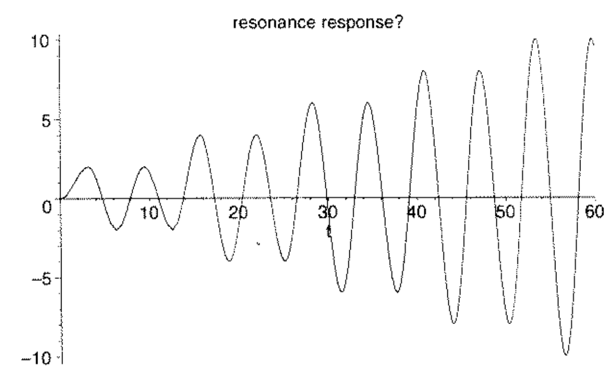
```
> plot(k(t),t=0..60,color=black,title='forcing function, period =
  ?');
>
```




```
> w:=t->int(sin(t-tau)*k(tau),tau=0..t);
```

$$w := t \to \int_0^t \sin(t-\tau)\,\mathrm{k}(\tau)\,d\tau$$

```
> plot(w(t),t=0..60,color=black,title='resonance response?');
```



Hey, what happened????

(2a) : $\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$

$$= \int_0^\infty e^{-st}\, u(t-a) f(t-a)\, dt$$

$$= \int_0^a + \int_a^\infty e^{-st}\cdot 1\cdot f(t-a)\,dt = \int_0^\infty e^{-s(\tilde t + a)} f(\tilde t)\,d\tilde t = e^{-sa}\underbrace{\int_0^\infty e^{-s\tilde t} f(\tilde t)\,d\tilde t}_{F(s)}$$

$\int_0^a = 0$ since $u(t-a)=0$ here

$\tilde t = t-a$

$d\tilde t = dt$

■

<u>application</u>: (example + p. 477)

$$\begin{cases} x'' + 4x = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases} \qquad f(t) = \begin{cases} \cos 2t & 0 \le t \le 2\pi \\ 0 & t > 2\pi \end{cases}$$

forcing term begins to cause resonance, but then is turned off.

notice, $f(t) = (\cos 2t)\left[1 - u(t-2\pi)\right]$

$$= \cos 2t - \cos(2(t-2\pi))\,u(t-2\pi)$$

trickery.

$\mathcal{L}$: $s^2 X(s) + 4X(s) = \dfrac{s}{s^2+4} - e^{-2s\pi}\dfrac{s}{s^2+4}$

$$X(s) = \frac{s}{(s^2+4)^2} - e^{-2s\pi}\frac{s}{(s^2+4)^2}$$

table: $x(t) = \frac{t}{4}\sin 2t - u(t-2\pi)\left[\frac{(t-2\pi)}{4}\underbrace{\sin(2(t-2\pi))}_{\sin 2t}\right]$

$$= \frac{t}{4}\sin 2t\left[1 - u(t-2\pi)\right] + \frac{\pi}{2}u(t-2\pi)\sin 2t$$

$$\boxed{x(t) = \begin{cases} \frac{t}{4}\sin 2t & 0 \le t \le 2\pi \\ \frac{\pi}{2}\sin 2t & t > 2\pi \end{cases}}$$

note: you could also do this problem in pieces:

for $0 \le t \le 2\pi$ $x(t)$ is solving

$$\begin{cases} x'' + 4x = \cos 2t \\ x_0 = v_0 = 0 \end{cases}$$

so $X(s)(s^2+4) = \dfrac{s}{s^2+4}$

$$X(s) = \frac{s}{(s^2+4)^2}$$

table: $x(t) = \frac{t}{4} \sin 2t$

so at $t = 2\pi$, $x(2\pi) = 0$

$x'(2\pi) = \frac{2\pi}{4} \cdot 2 = \pi$

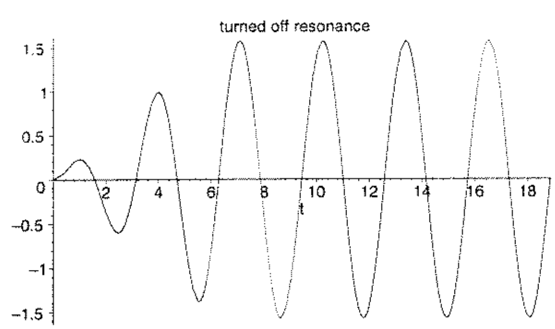
then for $t \geq 2\pi$ $x(t)$ solves

$$\begin{cases} x'' + 4x = 0 \\ x(2\pi) = 0 \\ x'(2\pi) = \pi \end{cases}$$

so $x(t) = A \cos 2t + B \sin 2t$

$= \frac{\pi}{2} \sin 2t$ ✓.

```
> plot(t/4*sin(2*t)*(1-Heaviside(t-2*Pi))
  +Pi/2*Heaviside(t-2*Pi)*sin(2*t),t=0..6*Pi, color=black,
  title='turned off resonance');
```




```
>
```