Math 2280-2
Maple Project 1, Part 1.
January 23, 2001

This project is available on-line at our course Maple page,
http://www.math.utah.edu/~korevaar/2280spring01/2280maple.html   You can also download the
answer template from that location. If you are shaky about Maple or the computer-lab, there is a tutorial
(really written for Math 2250 students) which you can also download and work on, available from the
same location. Always make sure that you download the ".mws" files to open from the Maple window.
If you want versions to print out or to look at from your browser, try the ".pdf" files as well.
    This project has two parts. In Part 1 you will study logistic equation applications from the computer
project of section 2.1 in the text, *Differential Equations and Boundary Value Problems*, second edition,
by Edwards-Penney.   In Part 2 of your Maple project you will study numerical methods for the solution
of first order differential equations, using ideas from sections 2.4-2.6.   Each part of the project has a
solution template into which you will do your work. These templates are also available at the course
Maple page.
    **It is very important when you read Math in the text, here, or elsewhere, that you read slowly,
sentence by sentence, making sure to think about and grasp each concept before moving on to the
next one. If you try to read mathematics as quickly as you read the newspaper you will become
lost very quickly.**

Part 1: Recall the logistic equation for population change which you have just been studying:

$$\frac{dP}{dt} = kP\,(M - P)$$

or, following the text and its equation numbering at page 86:

$$eqtn1 := \frac{dP}{dt} = a\,P + b\,P^2$$

with a=kM and b=-k. In our model the parameters k,M,a,b were related to assumptions about birth and
death rates. Suppose you have a real population and want to pick parameters a and b to make a good
model. One way would be to try to estimate fertility and morbidity rates based on birth and death data,
but that could get quite complicated. For example, if you want to develop an accurate model of world
population growth based on this sort of analysis you would probably need to collect data from different
regions of our planet and develop different parameters for different societies, solve the problem in each
part of the globe and then add your results together.   A more simple-minded approach is to just assume
the logistic model and to use existing population data to deduce good choices for a and b. The book
explains a good way to do this on pages 86-87, which also checks along the way whether the model
seems to be justified by the data:
    If you divide the logistic DE, eqtn1 above, by P, you get

$$eqtn2 := \left[\frac{1}{P}\right]\left[\frac{dP}{dt}\right] = a + b\,P$$

If you have multi-year population data you can get good estimates for the left side of eqtn2 by using
difference quotients to estimate dP/dt. Dividing by P gives an estimate for (dP/dt)/P. If you can do this
for a lot of different times you can get a collection of points [P, (1/P)(dP/dt)]. If these points seem to lie

approximately along a line, then eqtn2 will be a good model for the population problem, and you can estimate the parameters "a" and "b" by getting the vertical axis-intercept and slope of the line which best fits the point data, respectively.

For example, looking at the USA data in Figure 2.1.7 on page 87, we can estimate dP/dt in 1800 in a "centered" way by taking the difference (P(1810)-P(1790))/20 (in units of people/year). Centered differences as in (eqtn3) page 86, generally give more accurate estimates for the derivative than the one-sided differences used in the limit definition. This is shown geometrically in Figure 2.1.6, page 87. Finally we would divide our centered estimate for dP/dt in 1800 by P(1800) to get an estimate for (dP/dt)/P in 1800, for the USA population:

```
> restart:
>
  P1:=5.308;
     #population in 1800, in millions.
     #Remember you can do multiline commands
     #by holding down the shift key when you hit return
     #or enter.
  P1prime:=(7.240-3.929)/20;
     #estimate for dP/dt in 1800, in millions of
     #people per year, see page 86 bottom
     #and top right entry in Figure 2.1.7 page 87
```

$$P1 := 5.308$$

$$P1prime := .1655500000$$

```
> P1primeoverP1:=P1prime/P1;
     #estimate for (1/P)*(dP/dt) in 1800;
```

$$P1primeoverP1 := .03118877167$$

```
> point1:=[P1,P1primeoverP1];
     #the left-most point on the graph of Figure 2.1.8
```

$$point1 := [5.308, .03118877167]$$

```
>
```

We can automate this process. We are still using the table on page 87, and the one on pate 80. You should verify how these numbers were extracted from the tables.

```
> with(linalg):
 Warning, the protected names norm and trace have been redefined and unprotected

> pops:=matrix(21,2,[[1790,3.9],[1800,5.3],[1810,7.2],
           [1820,9.6],[1830,12.9],[1840,17.1],
           [1850,23.2],[1860,31.4],[1870,38.6],
           [1880,50.2],[1890,63.0],[1900,76.2],
           [1910,92.2],[1920,106.0],[1930,123.2],
           [1940,132.2],[1950,151.3],[1960,179.3],
           [1970,203.3],[1980,225.6],[1990,248.7]]);
                #matrix of populations


>
```

$$pops := \begin{bmatrix} 1790 & 3.9 \\ 1800 & 5.3 \\ 1810 & 7.2 \\ 1820 & 9.6 \\ 1830 & 12.9 \\ 1840 & 17.1 \\ 1850 & 23.2 \\ 1860 & 31.4 \\ 1870 & 38.6 \\ 1880 & 50.2 \\ 1890 & 63.0 \\ 1900 & 76.2 \\ 1910 & 92.2 \\ 1920 & 106.0 \\ 1930 & 123.2 \\ 1940 & 132.2 \\ 1950 & 151.3 \\ 1960 & 179.3 \\ 1970 & 203.3 \\ 1980 & 225.6 \\ 1990 & 248.7 \end{bmatrix}$$

```
> for i from 1 to 11 do
  lspoints[i]:=[pops[i+1,2],
     (pops[i+2,2]-pops[i,2])/(20*pops[i+1,2])];
  od;
>
```

$$lspoints_1 := [5.3, .03113207547]$$

$$lspoints_2 := [7.2, .02986111111]$$

$$lspoints_3 := [9.6, .02968750000]$$

$$lspoints_4 := [12.9, .02906976744]$$

$$lspoints_5 := [17.1, .03011695906]$$

$$lspoints_6 := [23.2, .03081896552]$$

$$lspoints_7 := [31.4, .02452229300]$$

$$lspoints_8 := [38.6, .02435233160]$$

$$lspoints_9 := [50.2, .02430278884]$$

$$lspoints_{10} := [63.0, .02063492064]$$

$$lspoints_{11} := [76.2, .01916010498]$$

We will follow investigation A and find the least squares line fit to this data, in order to extract values for "a" and "b" in eqtn2. We remember how to do this from Math 2270, section 8.4 of Kolman.

```
> A:=matrix(11,2);
  B:=vector(11);
```

$$A := \text{array}(1 .. 11, 1 .. 2, [\ ])$$

$$B := \text{array}(1 .. 11, [\ ])$$

```
> for i from 1 to 11 do
  A[i,1]:=lspoints[i][1]:
  A[i,2]:=1:
  B[i]:=lspoints[i][2]:
  od:
```

```
> evalm(A);evalm(B);
     #check work
```

$$\begin{bmatrix} 5.3 & 1 \\ 7.2 & 1 \\ 9.6 & 1 \\ 12.9 & 1 \\ 17.1 & 1 \\ 23.2 & 1 \\ 31.4 & 1 \\ 38.6 & 1 \\ 50.2 & 1 \\ 63.0 & 1 \\ 76.2 & 1 \end{bmatrix}$$

[.03113207547, .02986111111, .02968750000, .02906976744, .03011695906, .03081896552,

```
       .02452229300, .02435233160, .02430278884, .02063492064, .01916010498]
> linsolve(transpose(A)&*(A),transpose(A)&*B);
      #least squares solution


                          [-.0001694136800, .03185105239]
> a:=.3185105239e-1;
       #intercept
   b:=-.1694136800e-3;
       #slope
```

$$a := .03185105239$$
$$b := -.0001694136800$$

Now we can create figure 2.1.8:

```
> with(plots): #load the plotting package
 Warning, the name changecoords has been redefined


> pict1:=pointplot({seq(
      lspoints[i],i=1..11)}):
       #a plot of the points,
       #with output suppressed.  Make sure
       #to end this command with a colon! If you use a
       #semicolon you get a huge mess when you have
       #a lot of points
> line:=plot(a+b*P,P=0..100, color=black):   #same warning here
       #about using a colon vs semicolon
> display({pict1,line}, title="Figure 2.1.8");
       #now use a semicolon, and
       #get a picture containing the line and the
       #three points we computed.
```
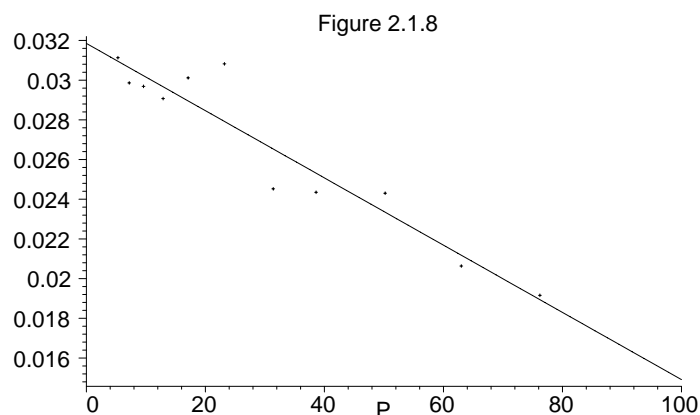

Figure 2.1.8

Figure 2.1.8 shows that the logistic model is a pretty good one for the U.S. population in the 1800's.
Let's use the "a" and "b" which we found with the least squares fit, use the population in 1900 for our
initial condition, and see how the logistic model works when we try to extend it to the 1900's:
```
> with(DEtools):   #load the DE package
 Warning, the name adjoint has been redefined
```

```
> deqtn1:=diff(x(t),t)=a*x(t) + b*x(t)^2;
        #logistic eqtn with our parameters
```

$$deqtn1 := \frac{\partial}{\partial t} \, \mathrm{x}(t) = .03185105239 \, \mathrm{x}(t) - .0001694136800 \, \mathrm{x}(t)^2$$

```
> P:=dsolve({deqtn1,x(0)=76.21},x(t));
        #take 1900 as t=0 and solve the initial value
        #problem
```

$$P := \mathrm{x}(t) = 3185105239 \, \frac{1}{16941368 + \dfrac{189400358372}{7621} \, \mathbf{e}^{\left(-\frac{3185105239}{100000000000} t\right)}}$$

```
> f:=s->evalf(subs(t=s,rhs(P))); #extract the right-hand side
        #from the above expression to make your solution function
```
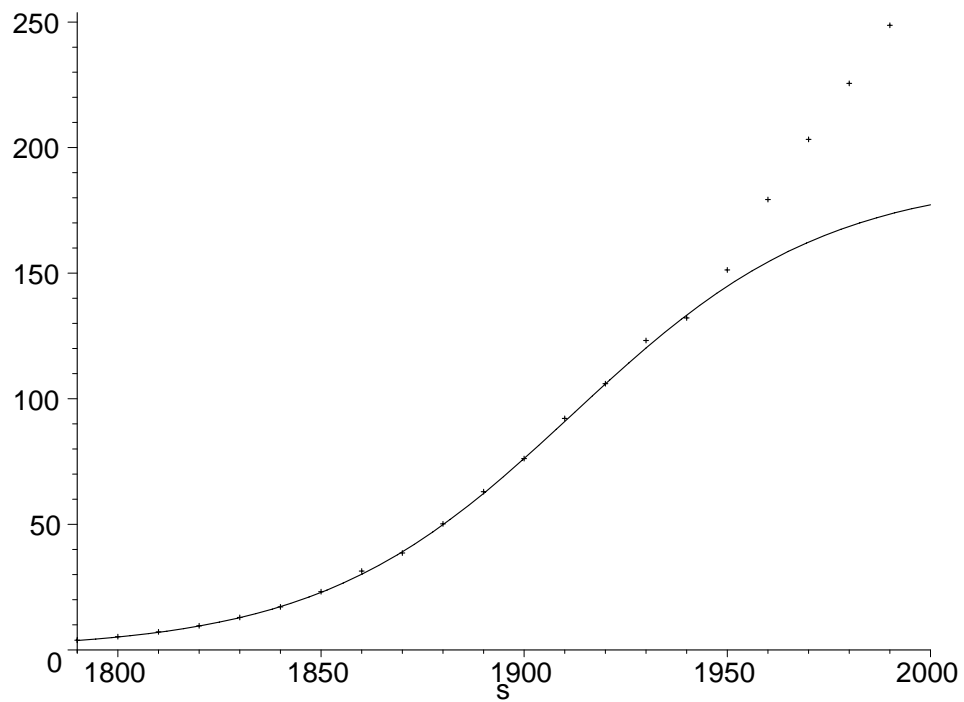
$$f := s \rightarrow \mathrm{evalf}(\mathrm{subs}(t = s, \, \mathrm{rhs}(P)))$$

```
> f(s);     #check that those weird subs and rhs commands really work
```

$$.3185105239 \; 10^{10} \, \frac{1}{.16941368 \; 10^8 + .2485242860 \; 10^8 \, \mathbf{e}^{(-.03185105239 \, s)}}$$

We can see how the model works by plotting actual populations against predicted ones, for the 1900's.

```
> actual:=pointplot({seq([pops[i,1],pops[i,2]],i=1..21)}):
  model:=plot(f(s-1900),s=1790..2000,color=black):
  display({actual,model});
```

Can you think of possible reasons why the model began to fail so badly around 1950? There are several.

Your first Maple project is to carry out an anlysis like this, for the world poplulation. This is Investigation C on page 88.