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 I.D. number.....

Math 2280-2
PRACTICE EXAM

April 2, 2001

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** On the real midterm there will only be 100 points possible, and the point values for each problem will be indicated in the right-hand margin. (This practice exam may be somewhat longer than the real thing.)
Good Luck!

1) Consider the following two-tank configuration. In tank one there is uniformly mixed volume of 200 gallons, and pounds of solute $x(t)$. In tank two there is mixed volume of 100 gallons and pounds of solute $y(t)$. Water is pumped into tank one at a constant rate of 10 gallons/minute from an outside source, and this water has a constant solute concentration of 3 pounds/gallon. Water is pumped from tank one to tank two at constant rate of 10 gallons/minute, and from tank two to the sewer, also at a rate of 10 gallons/minute. Initially the water in each tank is pure.

1a) Derive the system of first order differential equations which governs the process described above. (10 points)

1b) The homogenous part of the system of differential equations above is

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{20} & 0 \\ \frac{1}{20} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a fundamental matrix solution for this homogenous system. (10 points)

1c) Find the matrix exponential for the matrix in part 1b). (10 points)

1d) Find a particular solution for the inhomogenous system in part 1a.

(10 points)

1e) Solve the initial value problem in part 1a

(10 points)

2) DERIVE either the variation of parameters formula using a general FMS, or the particular case of it using matrix exponentials, for finding solutions to the inhomogeneous system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = Ax + f(t)$$

(15 points)

3) Consider the following configuration of springs, with positive displacements from equilibrium measured to the right, as indicated.

3a) Derive the system of second order differential equations which models this system. Assume that there are no external forces.

(5 points)

3b) Assume that in appropriate units $m_1=2$, $m_2=2$, $k_1=4$, $k_2=6$. Show that in this case your system above reduces to

$$\begin{bmatrix} \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2x + 2y \\ 2x - 5y \end{bmatrix}$$

(5 points)

3c) Find the general solution to the unforced system (7b).

(15 points)

3d) Assuming ω is not a natural frequency for the problem above, find a particular solution to the forced system

$$\begin{bmatrix} \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2x + 2y + \cos(\omega t) \\ 2x - 5y - \cos(\omega t) \end{bmatrix}$$

(10 points)

4) Consider the following system of differential equations which is supposed to model two interacting species:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 5x - x^2 - xy \\ -2y + xy \end{bmatrix}$$

4a) Would this system be modeling a cooperative, competitive, or predator-prey situation. Explain. (5 points)

4b) Show that there are three equilibrium solutions to this system, namely [0,0], [5,0], [2,3]. (5 points)

4c) Compute the linearized differential equation near each of the three equilibria from part (a). For each equilibrium solution use eigenvalue, eigenvector analysis to sketch a local phase portrait near the equilibrium solution, and indicate what type of equilibrium you are dealing with, and its stability characteristics. (45 points)

4d) Sketch the phase portrait (in the first quadrant) for the full non-linear system, using your information from part 4c. Explain what this means for the long-time behavior of solutions to this system. (15 points)

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