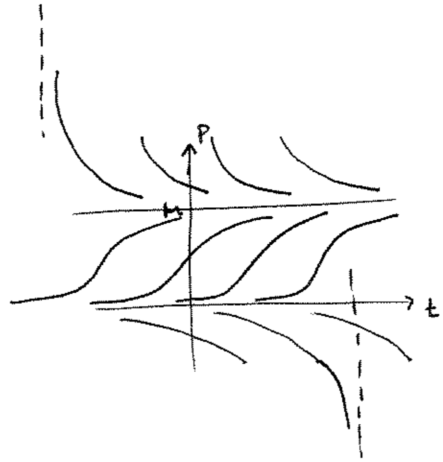
Math 2280-1
Tuesday Sept 9.

- Yesterday we solved the logistic IVP:

$$\text{IVP}\begin{cases} \frac{dP}{dt} = kP(M-P) \\ P(0) = P_0 \end{cases}$$



$$\text{soltn: } P(t) = \frac{M}{1 + \left(\frac{M-P_0}{P_0}\right)e^{-Mkt}}$$

$$= \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}$$

if you set denom = 0

$$P_0 = (P_0 - M)e^{-Mkt}$$

$$\frac{P_0}{P_0 - M} = e^{-Mkt}$$

if $P_0 > 0$ then for $0 < P_0 < M$ there are no t-values making eqtn true, since LHS is negative

if $P_0 > M$ there is a negative t-value to making eqtn true, $\lim_{t \to t_0^+} P(t) = +\infty$.

if $P_0 < 0$ there is a positive t-value to making eqtn true, $\lim_{t \to t_0^-} P(t) = -\infty$

- Go over U.S. population modeling from Monday handout.

Before discussing more population (and also velocity-acceleration) models, let's talk about the general language that gets used...

§2.2: A general 1^{st} order DE $\frac{dx}{dt} = f(t,x)$

is called autonomous if $\frac{dx}{dt} = f(x)$ ($\frac{dx}{dt}$ only depends on x itself, not also on t)

def $x(t)$ is an equilibrium sol'n to a DE iff $x(t) \equiv C$, a constant

If the DE is autonomous and $x(t) \equiv C$ is an equilibrium sol'n, then
$$0 = \frac{dx}{dt} = f(x) = f(C) = 0.$$

And if $f(C) = 0$ then $x(t) \equiv C$ is an equilibrium sol'n.

> equilibrium sol's of $\frac{dx}{dt} = f(c)$ are exactly the fcns $x(t) \equiv c$ where $f(c) = 0$

example

$$\frac{dx}{dt} = kx(M-x)$$

$x \equiv 0$, $x \equiv M$ are the equil. sol'ns

example find the equil. sol'ns of

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

Def Let c be an equilibrium sol'n for a deqtn. Then

c is <u>stable</u> iff $\forall \varepsilon > 0 \ \exists \ \delta > 0$ s.t. for sol'ns $x(t)$ with $|x(0) - c| < \delta$ we have $|x(t) - c| < \varepsilon \quad \forall \ t > 0$

(sol'ns which start close enough to c stay arbitrarily close to it.)

c is <u>unstable</u> if it is not stable

<u>example</u>: Make the slope field (or alternately, sketch a sufficient sample of sol'n graphs) and phase portrait for

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

and discuss stability of the equil sol'ns $x = 0, x = -1$

<u>example</u> find equilibria and discuss stability for $\frac{dx}{dt} = 3x - x^2$

Def c is called <u>asymptotically stable</u> (stronger requirement than stable) iff $\exists \ \delta > 0$ s.t. $|x(t) - c| < \delta \Rightarrow \lim_{t \to \infty} x(t) = c$.

Are any of the equilibria you found above asymptotically stable?

<u>Theorem</u>: Consider $\frac{dx}{dt} = f(x)$, f continuously differentiable.

if $f(c) = 0$ then $f'(c) > 0 \Rightarrow$ $\underset{c}{- - - \ 0 + + +}$ sign f $\Rightarrow$ $\leftarrow \underset{c}{|} \rightarrow$ unstable

f increasing

$f'(c) < 0 \Rightarrow$ $\underset{c}{+ + + \ 0 - - -}$ sign f $\Rightarrow$ $\rightarrow \underset{c}{|} \leftarrow$ stable

f decreasing

$f'(c) = 0$ You must do additional work.

More "real" examples

doomsday extinction:

$$\frac{dP}{dt} = aP^2 - bP \qquad a, b > 0.$$

$$= kP(P-M) \qquad \begin{array}{l} k = a \\ -kM = b \end{array}$$

e.g. if $\beta(t)$ (fertility rate) $\overset{\text{proportional to}}{\downarrow} \sim P$

(e.g. herd roams in a fixed area and mates randomly upon meeting opposite-sex mate)

then $B(t) = aP^2$;

perhaps $D(t) = -bP$.

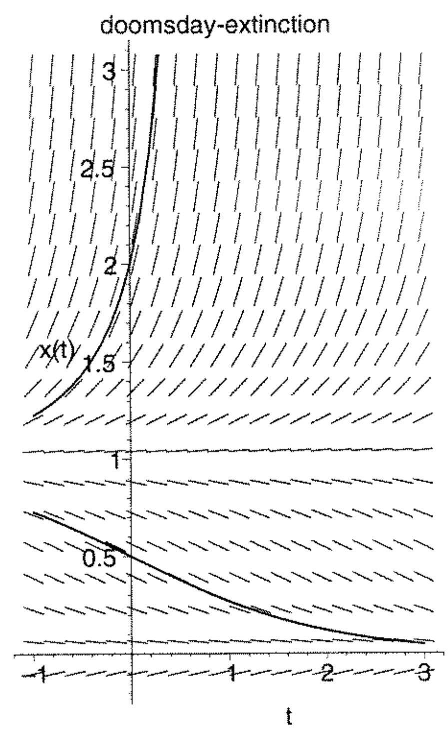
$+ + +\ 0 - - - - 0 + +$ sign(f(P))

$0 \qquad M$

0 stable; M Unstable!

If $0 < P_0 < M$ then $\lim_{t \to \infty} P(t) = 0$ (so in real life, extinction)

If $P_0 > M$ then $\exists\, t_1 < \infty$ s.t. $\lim_{t \to t_1} P(t) = +\infty$ (doomsday!)



<u>example</u>

$$\begin{cases} \frac{dx}{dt} = x(x-1) \\ x(0) = 2 \end{cases}$$

Find the time of doomsday.

(ans $t = \ln 2$!)