Math 2280-1 & 2250-1
Monday Sept 8

§2.1 Improved population models

Let $P(t)$ = population (e.g. # people, # fish, etc.)
$B(t)$ = birth rate (e.g. people/year)
$D(t)$ = death rate (e.g. people/year)

$\beta(t) = \frac{B(t)}{P(t)}$ (people/year)/people e.g. .1 person/year per person.
fertility rate

$\delta(t) = \frac{D(t)}{P(t)}$ morbidity rate

Then
$$\frac{dP}{dt} = B(t) - D(t)$$

$$\boxed{\frac{\left(\frac{dP}{dt}\right)}{P} = \beta(t) - \delta(t)}$$

model 1 : $\beta(t) \equiv \beta$ const, $\delta(t) \equiv \delta$ const $\Rightarrow$ $\frac{dP}{dt} = (\beta - \delta)P = kP$

either exponential growth or decay, depending on sign of $k$
... we know this DE!

model 2 : $\beta = \beta_0 - \beta_1 P$ ($\beta_0, \beta_1$ const)
$\delta = \delta_0 + \delta_1 P$

$\beta_1 > 0$ "sophisticated" population *
$\delta_1 > 0$ ~ finite resources, disease, fighting

So
$$\frac{dP}{dt} = (\beta - \delta)P$$
$$= [(\beta_0 - \beta_1 P) - (\delta_0 + \delta_1 P)]\,P$$
$$= [(\beta_0 - \delta_0) - (\beta_1 + \delta_1)P]\,P$$

$$\boxed{\begin{aligned}\frac{dP}{dt} &= kP(M-P)\\ \frac{dP}{dt} &= aP - bP^2\end{aligned}}$$

$k = \beta_1 + \delta_1$ , $M = \frac{\beta_0 - \delta_0}{\beta_1 + \delta_1}$

$a = kM$, $b = k$ $\left(\begin{aligned}k &= b\\ M &= a/b\end{aligned}\right)$

Logistic eqtn

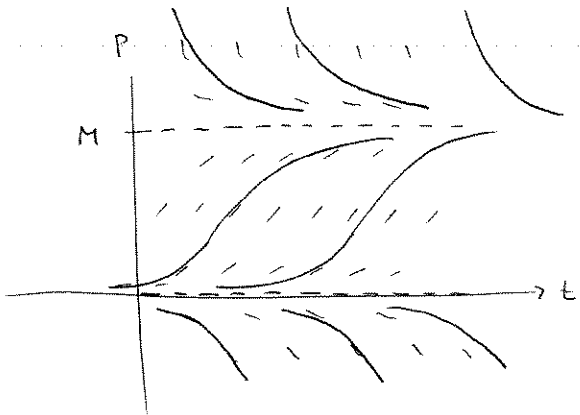
(There is a model 3 we'll discuss Monday for which $\beta = \beta_0 + \beta_1 P$ ("alligators") which can lead to the "doomsday-extinction" de, $\frac{dP}{dt} = -aP + bP^2$.)

What does the slope field predict for sol'ns to

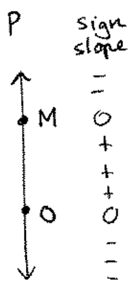
$$\text{IVP} \begin{cases} \dfrac{dP}{dt} = kP(M-P) & (k, M > 0) \\ P(0) = P_0 \end{cases}$$

notice isoclines are horizontal lines in the $(t, P)$ plane:

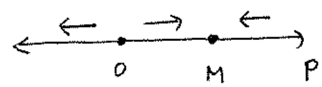
$P = 0 \Rightarrow m = 0$
$P = M \Rightarrow m = 0$




slope field picture




leads to phase portrait:




$P = 0$ and $P = M$ are called equilibrium solutions (because they are constant in time)

- It appears that any IVP sol'n $P(t)$ with $P(0) = P_0 > 0$ will converge to $P = M$.

$\boxed{M := \text{carrying capacity.}}$

$\boxed{\text{Solve the logistic IVP by separating variables}}$ (Use back of page if necessary!) (Then analyze solution to verify slope field & phase portrait predictions)

Sol: $P(t) = \dfrac{M}{\left(\frac{M - P_0}{P_0}\right) e^{-Mkt} + 1}$