Math 2280-1
Wednesday Sept. 3

- Derive Torricelli's Law for draining cisterns, and the separable DE it implies.
  - this is page 3 of Tuesday's notes. there are a couple of sign-errors on that page, which we will correct as we go through the notes. The final boxed separable DE is right.
- If T_μ is the time it takes water to drain from height h to height μh in a cylinder show $T_0 = \frac{T_\mu}{1-\sqrt{\mu}}$ (page 4 Tuesday)

- In our experiment yesterday,
  $T_{1/2} = 34.3$ sec
  $T_0 = 110.5$ sec.

  Using the value of $T_{1/2}$, what does our model predict for T_0?
  What was our relative error?
  Can you think of measurement, model errors which could contribute?

§1.5 Linear 1^{st} order DE's.

(1) $$\frac{dy}{dx} + P(x)y = Q(x)$$

Notice the left side of this eqtn, $L(y) := y' + P(x)y$ is <u>linear</u> :

$L(y_1+y_2) = L(y_1) + L(y_2)$ (y_1, y_2 diffble)

$L(cy) = cL(y)$ (c a const, y diffble)

Solution method is to multiply both sides by a non-zero fcn ("integrating factor") so that we can antidifferentiate wrt x to deduce $y(x)$:

eqtn (1) is equiv. to

(2) $$e^{\int P(x)dx}\left(y' + P(x)y\right) = e^{\int P(x)dx} Q(x)$$

where $\int P(x)dx$ is any particular antideriv. of $P(x)$

equiv to

(3) $$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = \underbrace{e^{\int P(x)dx} Q(x)}$$

or (antidifferentiating)

(4) $$e^{\int P(x)dx} y = \int [\quad] \, dx$$; divide by $e^{\int P(x)dx}$ to get $y(x)$

<u>Example</u> HW problem #6 §1.3 you are asked to plug in $y = x + Ce^{-x}$ to show it solves the DE

$$y' = x - y + 1$$

Use the algorithm above to <u>derive</u> the given soltns:

$$y' + 1y = x + 1$$

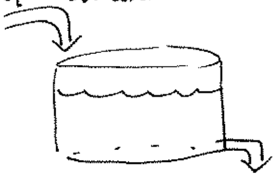
example 2 page 48-49:

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x$$

ans $y = 2 + C(x^2+1)^{-3/2}$

• Example: <u>mixing problems</u> (p. 51) applies also to pharmacology & environment

r_i = rate in ℓ/s
c_i = concentration in gm/ℓ



r_o = rate out ℓ/s
c_o = concentration out g/ℓ

$V(t)$ = volume in tank at time t (ℓ)
$x(t)$ = amount of solute in tank (gm)

$c(t) = \dfrac{x(t)}{V(t)}$ $\frac{g}{\ell}$ (average) concentration in tank

Assume mixture is uniform so $c(t)$ is spatially constant

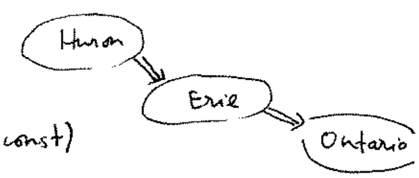
$$\frac{dV}{dt} = r_i - r_o \qquad \ell/s \longrightarrow V(t) = V_0 + \int_0^t r_i(s) - r_o(s)\,ds \qquad \S 1.2$$

$$\frac{dx}{dt} = r_i c_i - r_o c_o \qquad \frac{\ell}{s}\frac{g}{\ell} \quad g/s$$

$$= r_i c_i - r_o \frac{x(t)}{V(t)} \qquad \text{by "well mixed" model}$$

$$\boxed{\frac{dx}{dt} + \underbrace{\frac{r_o}{V(t)}}_{P(t)} x(t) = \underbrace{r_i c_i}_{Q(t)}}$$

<u>Example 4 p. 52</u> Lake Erie pollution



$V(t) = 480\ km^3$
$r = r_i = r_o = 350\ km^3/year$ (so V is const)
$c_i = c$ = pollutant conc of lake Huron
$x_0 = (5c)V$ (read problem).

$\boxed{\text{When will Erie pollution concentration have decreased to twice that of Huron?}}$

i.e. solve, and analyze

$$\begin{cases} \dfrac{dx}{dt} = r_i c_i - r_o c_o \\ x(0) = 5cV \end{cases}$$

$$\frac{dx}{dt} = rc - r\frac{x}{V}$$

for fun, notice this DE is also separable. Can you solve it that way?

$$\frac{dx}{dt} + \frac{r}{V}x = rc$$

$$\left(e^{\frac{r}{V}t}x\right)' = rc\,e^{\frac{r}{V}t}$$

$$e^{\frac{r}{V}t}x = rc\frac{V}{r}e^{\frac{r}{V}t} + C$$

$$x = cV + Ce^{-\frac{r}{V}t}$$

$$x(0) = 5cV \Rightarrow C = 4cV$$

$$x(t) = cV + 4cVe^{-\frac{r}{V}t}$$

set $x(T) = 2cV = cV + 4cVe^{-\frac{r}{V}T}$

$$.25 = e^{-\frac{r}{V}T}$$

$$-\frac{r}{V}T = \ln(.25)$$

$$T = -\ln(.25)\frac{V}{r}$$

$$= \ln(4)\frac{480}{350} \quad \text{years}$$

$$\boxed{\approx 1.90 \text{ years}}$$