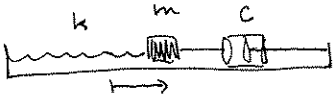
Math 2280-1
Monday Sept 29

§3.4 : Applications of const. coeff homogeneous DE's.
The spring, the pendulum.



spring, mass, dashpot configuration.

$x(t)$, displacement of mass from equilibrium

k = "Hooke's constant"
c = "coefficient of friction

Newton + linearization:

$$m x'' = -kx - cx'$$

$$\boxed{m x'' + c x' + k x = 0}$$

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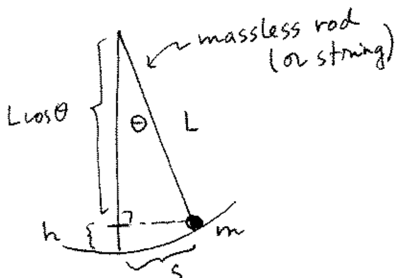
but very similar mathematically is the <u>pendulum equation</u>



$\theta(t)$

mass moves in a circular arc

$s(t) = L\theta(t)$ is (signed) distance to "bottom

$v(t) = s'(t) = L\theta'(t)$

neglecting drag this a conservative system

so total energy = KE + PE = constant

$$= \tfrac{1}{2} m \left(L\theta'(t)\right)^2 + mgh$$

$$= \tfrac{1}{2} m L^2 \theta'(t)^2 + mgL(1-\cos\theta)$$

$$\frac{d}{dt}(KE + PE) \equiv 0$$

$$\Rightarrow mL\left(\theta'\theta'' L + g\sin\theta\,\theta'\right) \equiv 0$$

$$mL\theta'\underbrace{\left(L\theta'' + g\sin\theta\right)}_{\text{so } = 0.} \equiv 0$$

only zero at isolated instances

Linearize! (near $\theta = 0$)
$\sin\theta \approx \theta$

$$\boxed{\begin{gathered} L\theta'' + g\theta = 0 \\ \theta'' + \tfrac{g}{L}\theta = 0 \end{gathered}}$$

(We could add drag and get $L\theta'' + c\theta' + g\theta = 0$, "same" DE as for mass-spring!)

<u>Case 1</u> free undamped motion

$$m x'' + k x = 0 \qquad (\text{or } L\theta'' + g\theta = 0)$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0 \qquad \left(\omega_0 := \sqrt{\tfrac{k}{m}}\right)$$

for $x(t) = e^{rt}$,

$$L(x) = (r^2 + \omega_0^2)e^{rt} = 0 \text{ iff } r = \pm i\omega_0$$

$$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$$

so

$$x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$= C\cos(\omega_0 t - \alpha)$$

$x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \alpha)$

$= C\cos(\omega_0(t-\delta))$

$C :=$ amplitude

$\alpha :=$ phase ; $\delta = \alpha/\omega_0 =$ "lag"

$\omega_0 :=$ angular frequency (rad/sec)

$\nu = f :=$ frequency (cycles/sec or herz)

$= \frac{\omega_0}{2\pi}$

$T :=$ period (secs/cycle) $= \frac{1}{f} = \frac{2\pi}{\omega_0}$

simple harmonic motion

$C\cos(\omega_0 t - \alpha)$

$= C\cos\omega_0 t\cos\alpha + C\sin\omega_0 t\sin\alpha$

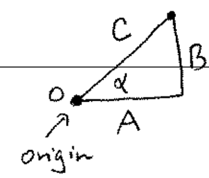
$= A\cos\omega_0 t + B\sin\omega_0 t$

Thus

$C\cos\alpha = A$
$C\sin\alpha = B$ so $C^2 = A^2 + B^2$

$\cos\alpha = A/C$
$\sin\alpha = B/C$

summarized by



depending on the signs of A&B the hypotenuse can point into any quadrant!

if $-\pi/2 < \alpha < \pi/2$,
$\alpha = \arctan(B/A)$

if $0 < \alpha < \pi$
$\alpha = \arccos(A/C)$

if $-\pi/2 < \alpha < \pi/2$
$\alpha = \arcsin(B/C)$

in third quad $(\pi < \alpha < 3\pi/2)$,
$\alpha = \pi + \arctan B/A$ works.

Example 1 page 189

$m = \frac{1}{2}$ kg

$2k = 100$ N so $k = 50$ N/m

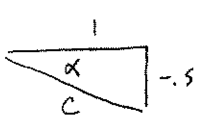
$\frac{1}{2}x'' + 50x = 0$

$$\begin{cases} x'' + 100x = 0 \\ x_0 = 1 \text{ m} \\ v_0 = -5 \text{ m/s} \end{cases}$$

$x_H(t) = A\cos 10t + B\sin 10t$

use IV's to find A & B:

ans: $x_H(t) = \cos 10t - .5\sin 10t$



$C = \sqrt{1.25} \approx 1.118$

$\alpha = \arctan(-.5) \approx -.464$

$x_H(t) \approx 1.118\cos(10t + .464)$ (our ans) $= 1.118\cos(10(t + .0464))$

$\approx 1.118\cos(10t - 5.8195)$ (book ans)

These are the same $x_H(t)$!

Math 2280-1
Simple harmonic motion
Example 1 page 189

```
> x:=t->cos(10*t)-.5*sin(10*t);
```

$$x := t \to \cos(10\,t) - 0.5\,\sin(10\,t)$$

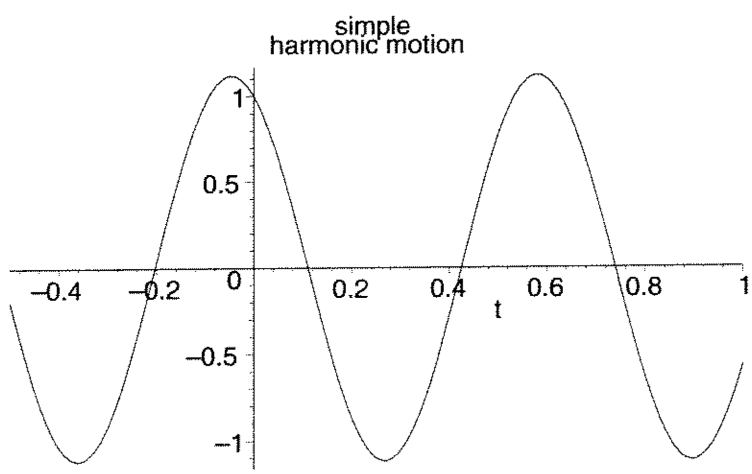
```
> C:=sqrt(1.25);  #amplitude
  alpha:=arctan(-.5);  #phase
  alpha+evalf(2*Pi); #book's phase
  delta:=alpha/10; #time lag
```

$$C := 1.118033989$$
$$\alpha := -0.4636476090$$
$$5.819537699$$
$$\delta := -0.04636476090$$

```
> y:=t->C*cos(10*t-alpha); #same function!
  omega:=10:  #angular frequency
  nu:=evalf(10/(2*Pi));  #frequency
  T:=1/nu;  #period
>
```

$$y := t \to C\cos(10\,t - \alpha)$$
$$\nu := 1.591549430$$
$$T := 0.6283185311$$

```
> with(plots):
  plot({x(t),y(t)},t=-.5..1,color=black,title=`simple
  harmonic motion`);
```



- now, add damping:

$$mx'' + cx' + kx = 0$$

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$$

$$x'' + 2px' + \omega_0^2 x = 0$$

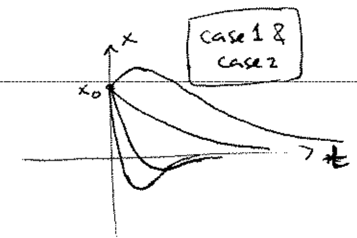
$\omega_0 = \sqrt{\frac{k}{m}}$ (as before), = undamped angular frequency (rad/time)

$p = \frac{c}{2m}$

e^{rt}: $\quad r^2 + 2pr + \omega_0^2 = 0$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$



Case 1 $\quad p^2 > \omega_0^2$: $\quad -p - \sqrt{p^2-\omega_0^2} < -p + \sqrt{p^2-\omega_0^2} < 0$

$$r_2 < r_1 < 0$$

$$x_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$= e^{r_1 t}\left[C_1 + c_2 e^{(r_2 - r_1)t}\right]$$

OVERDAMPED

at most one root & decays exponentially in time (Figure 3.4.7 p 188)

Case 2 $\quad p^2 = \omega_0^2$: double root $r = -p$

$$x_H(t) = c_1 e^{-pt} + c_2 t e^{-pt}$$
$$= e^{-pt}(c_1 + c_2 t) \quad \text{also}$$

; Figure 3.4.8.

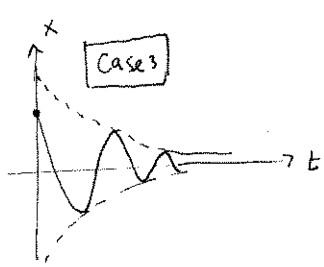
CRITICALLY DAMPED

Case 3 $\quad p^2 < \omega_0^2$, $\quad r = -p \pm i\underbrace{\sqrt{\omega_0^2 - p^2}}_{\omega_1 := \sqrt{\omega_0^2 - p^2}}$

$$x_H(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$$
$$= e^{-pt}(C\cos(\omega_1 t - \alpha))$$

UNDERDAMPED

decays exponentially but oscillates ∞'ly often.



Ce^{-pt} is called "time-varying amplitude"
ω_1 is called "pseudo angular frequency"
etc.

notice $\omega_1 < \omega_0$, i.e. damping retards the oscillation.

Example 2 p. 192

$$\begin{cases} x'' + 2x' + 100x = 0 \\ x_0 = 1 \text{ m} \\ v_0 = -5 \text{ m/s} \end{cases}$$

contrasted to Example 1

$$\begin{cases} x'' + 100x = 0 \\ x_0 = 1 \\ v_0 = -5 \end{cases}$$

sol'n e^{rt}: $r^2 + 2r + 100 = 0$

$$(r+1)^2 + 99 = 0$$

$$(r+1+i\sqrt{99})(r+1-i\sqrt{99}) = 0$$

$$r = -1 \pm i\sqrt{99}$$

$$x_H(t) = e^{-t}\left(A\cos\sqrt{99}\,t + B\sin\sqrt{99}\,t\right)$$

$\omega_1 = \sqrt{99} < \omega_0 = 10$, because of damping

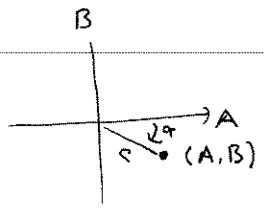
$$x(0) = \boxed{1 = A}$$

$$v(0) = -5 \Rightarrow -5 = -A + \omega_1 B$$

$$\underset{\overset{\shortparallel}{-4}}{A - 5} = \omega_1 B \qquad \boxed{B = \frac{-4}{\sqrt{99}}}$$

$$C_1 = \sqrt{A^2 + B^2} = \sqrt{1 + \tfrac{16}{99}} \approx 1.078$$

$$\alpha_1 = \arctan\left(-\tfrac{4}{\sqrt{99}}\right) \approx -.382 \text{ rad.}$$



Damping added to Example 1, i.e.
Example 2 page 192

```
> x1:=t->exp(-t)*(cos(sqrt(99)*t)-4/sqrt(99)*sin(sqrt(99)*t));
```

$$x1 := t \rightarrow \mathbf{e}^{(-t)}\left(\cos(\sqrt{99}\,t) - \frac{4\sin(\sqrt{99}\,t)}{\sqrt{99}}\right)$$

```
> C1:=sqrt(1+16/99.);  #amplitude
  alpha1:=arctan(-4/sqrt(99.));  #phase
  alpha1+evalf(2*Pi); #book's phase
  omega1:=sqrt(99.):  #pseudo angular frequency
  nu1:=evalf(omega1/(2*Pi));  #pseudofrequency
  T1:=1/nu1;  #pseudoperiod
  delta1:=alpha1/omega1; #pseudolag
```

$$C1 := 1.077782985$$

$$\alpha 1 := -0.3822423467$$

$$5.900942961$$

$$\nu 1 := 1.583571689$$

$$T1 := 0.6314838835$$

$$\delta 1 := -0.03841680130$$

```
> y1:=t->C1*exp(-t)*cos(omega1*t-alpha1): #same function!
> plot({x1(t),y1(t),x(t),C1*exp(-t),-C1*exp(-t)},t=0..3,color=black,
  title='simple harmonic and underdamped
  motion');
```