Math 2280
Wednesday Sept 24

- Carefully go through the rest of Monday's notes, which explain how to find a basis for ker L when
$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 \qquad \left(a_j \text{ constant}\right) \quad (a_j \text{ real})$$
and the characteristic polynomial
$$p(r) := r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$$
has real roots only (either n distinct roots, or repeated roots.)

- With any leftover time, begin discussion of complex roots

Theorem 4 If r is a complex root of $p(r)$, $r = a+bi$ (a, b real) $i=\sqrt{-1}$

then $y_1 = e^{ax}\cos bx$
$y_2 = e^{ax}\sin bx$
are linearly independent soltns to $L(y)=0$.

proof: you could check this directly, but there's a more natural (and motivated) proof using complex exponentials.

Step 1 Euler's formula: $\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$ (θ real)

motivation: check Taylor series!

Step 2 $e^{\alpha + i\beta} := e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$ (α, β real)

so $e^{(a+ib)x} = e^{ax+ibx} = e^{ax}(\cos bx + i\sin bx)$ (x real, a, b real)

Step 3 $D_x(f(x) + ig(x)) := f'(x) + ig'(x)$ f, g real fcns

Step 4 $\quad D_x\left(e^{(a+ib)x}\right) = (a+ib)\,e^{(a+ib)x}$

i.e. if r is a complex number $\quad D_x\left(e^{rx}\right) = re^{rx}$.

This needs to be checked!!

<u>proof of Theorem 4</u> $\quad$ Let $r = a+bi$ a root of the characteristic polynomial p

Let $y = e^{rx} = e^{ax}(\cos bx) + i\,e^{ax}\sin bx$

$= y_1 + i\,y_2$

by step 4,
$L(y) = L(e^{rx}) = p(r)e^{rx} = 0.$

by Step 3, $L(y) = L(y_1 + iy_2) = L(y_1) + i\,L(y_2)$
(and linearity)

so $\overset{0+0i}{\overset{\shortparallel}{0}} = \underbrace{L(y_1)}_{\substack{\text{real} \\ \text{function,} \\ \text{since } a_i\text{'s} \\ \text{are real}}} + i\underbrace{L(y_2)}_{\substack{\text{imag. fcn} \\ \text{since } a_i\text{'s} \\ \text{are real.}}}$

Thus $L(y_1) = 0$ (equate real parts)
$L(y_2) = 0$ (equate imag. parts).

last step: Show $\begin{array}{l} y_1 = e^{ax}\cos bx \\ y_2 = e^{ax}\sin bx \end{array}$ are linearly independent:

<u>Exercise</u> Find the general solution of

$$y'' + 4y' + 5y = 0.$$