Math 2280-1
Monday Sept. 22

- finish example, page 3 Friday (we changed it to $L(y) = y''' + y'' - 30y'$)
- Then prove the theorem on page 2 Friday,
  that for $L := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$, $L: C^n(I) \to C(I)$
  $a_j(x) \in C(I)$
  then $\ker L = \{v \text{ s.t. } Lv = 0\}$ is n-dimensional
- then continue the page 3 Friday example:

Find <u>all</u> solutions to

$$y''' + y'' - 30y' = 56e^x$$

hint: $y = y_P + y_H$; try $y_P = Ce^x$.

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this will be our strategy to find the general solution to $L(y) = f(x)$:

1. Find the general sol'n to $L(y) = 0$, i.e. y_H
2. Find a particular sol'n y_P to $L(y_P) = f$
3. $y = y_P + y_H$

for the next few lectures we focus on solving $L(y) = 0$, for const. coeff. L.

Constant coefficient nth-order Linear operators L

i.e. $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y$ $\quad a_j = \text{const}$, $0 \le j \le n-1$

Def: For L as above, the polynomial $p(r) := r^n + a_{n-1}r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0$ is called the characteristic polynomial of L

Theorem 1: If the characteristic polynomial $p(r)$ of the n^{th} order const coeff linear operator L has n distinct roots $r_1, r_2, \ldots r_n$ (real roots) the a basis for $\ker(L)$ is

$$\{e^{r_1 x}, e^{r_2 x}, \ldots e^{r_n x}\}$$

i.e. $y_H = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$

proof: $L(e^{rx}) = p(r)e^{rx}$

so if $p(r_j) = 0$, $e^{r_j x}$ solves $L(y) = 0$

Since $\dim(\ker L) = n$, it suffices to show $\{e^{r_1 x}, e^{r_2 x}, \ldots e^{r_n x}\}$ are linearly ind!

(since then they span, since there are n of them.

suppose $\quad c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x} \equiv 0$

take derivs!!

Did you prove in 2270, that the Vandermonde Determinant:

$$V = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ r_1 & r_2 & r_3 & & r_n \\ r_1^2 & r_2^2 & r_3^2 & & r_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & r_3^{n-1} & & r_n^{n-1} \end{vmatrix}$$

satisfies

$$V = \prod_{i<j}(r_i - r_j) \neq 0 \text{ if } r_1, r_2, \ldots r_n \text{ are distinct ??}$$

<u>Theorem 2</u> Let $L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$ $\quad a_j$ const

So charact poly

$$p(r) = r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$$

<u>and</u> $L = D^{(n)} + a_{n-1} D^{(n-1)} + \dots + a_1 D + a_0$ $\quad$ (where D^k means $D \circ D \circ \dots \circ D$ k ~~two~~ times)

If $p(r) = (r-r_1)^{k} q(r)$ $\quad$ (degree $q = n-k$)

Then $\{e^{r_1 x}, x e^{r_1 x}, \dots x^{k-1} e^{r_1 x}\}$ are k lin. ind solutions to $L(y) = 0$.

<u>proof</u>:
$$\begin{aligned} L(y) &= p(D)(y) \\ &= q(D) \circ (D - r_1 I)^k (y) \\ &= q(D) \circ \left[(D - r_1 I)^k (y) \right] \end{aligned}$$

so $L(x^m e^{r_1 x}) = q(D) \circ \left[(D - r_1 I)^k (x^m e^{r_1 x}) \right]$

$= 0$ for $m \le k-1$

$\therefore$ $\frac{1}{2}$ ∎

$(D - r_1 I)(e^{r_1 x}) = r_1 e^{r_1 x} - r_1 e^{r_1 x} = 0$

$(D - r_1 I)(x e^{r_1 x}) = e^{r_1 x} + r_1 x e^{r_1 x} - r_1 x e^{r_1 x}$

$= e^{r_1 x}$; $\underline{(D - r_1 I)^2 (x e^{r_1 x}) = 0}$.

$(D - r_1 I)\, x^m e^{r_1 x} = \underset{\substack{\uparrow \\ \text{one lower power}}}{m x^{m-1}} e^{r_1 x}$

inductively,

$$(D - r_1 I)^{m+1} (x^m e^{r_1 x}) = 0$$

Notice $\{e^{r_1 x}, x e^{r_1 x}, \dots x^{k-1} e^{r_1 x}\}$ are lin ind on any interval, since

$$c_1 e^{r_1 x} + c_2 x e^{r_1 x} + \dots + c_{k-1} x^{k-1} e^{r_1 x} \equiv 0$$

iff $e^{r_1 x} \left[c_1 + c_2 x + \dots + c_{k-1} x^{k-1} \right] \equiv 0$

iff $c_1 + c_2 x + \dots + c_{k-1} x^{k-1} \equiv 0$

iff $c_1 = c_2 = \dots = c_{k-1} = 0$ since $\{1, x, x^2, \dots x^{k-1}\}$ are lin ind. (why?)

∎

<u>Theorem 3</u> If $p(r)$ factors,

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \dots (r - r_\ell)^{k_\ell} \qquad r_1, r_2, \dots r_\ell \text{ distinct}$$

then a basis for $L(y) = 0$ sol'ns is

$$\left\{ e^{r_1 x}, x e^{r_1 x}, \dots x^{k_1 - 1} e^{r_1 x}, e^{r_2 x}, x e^{r_2 x}, \dots x^{k_2 - 1} e^{r_2 x}, \dots \right\}.$$

<u>proof</u>: From thm 2, the y_j all satisfy $L(y_j) = 0$.
It is more work to show they are linearly independent (but true!)

<u>example</u>: Find the general sol'n to

$$y^{(4)} - 2y^{(2)} + y = 0$$