Math 2280-1
Friday Sept. 19

HW for Monday Sept 29
3.1 1, 2, 4, 5, 11, 13, 17 27, 29, 30, 31, 33, 34, 35
3.2 2 5 9 11 13 21 22 25 26
3.3 3, 10, 14 21, 22, 29, 33, 37
3.4 4, 5, 6 13, 15, 19, 23

finish pages 4-5 Wednesday....

recall, we already discussed theorems 1, 2.

<u>Theorem 1</u> $L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$ s.t. each a_j cont. on fixed interval I

then $L: C^n(I) \to C(I)$ is <u>linear</u>

$\{y(x) \mid y, y', .. y^{n} \text{ are cont. on } I\}$

$\{z(x) \text{ s.t. } z \text{ cont on } I\}$.

<u>Theorem 2</u> $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$, a_j cont on I $j = 0 \cdots n-1$

$x_0 \in I$, $f \in C(I)$, $\vec{b} \in \mathbb{R}^n$

then $\exists!$ sol'n to IVP, defined on all of I, i.e. $y \in C^{(n)}(I)$.

$$\text{IVP} \begin{cases} L(y) = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

- proof defered! -

<u>Theorem 3</u> $L: V \to W$ linear, $b \in W$.

then if $L(v_p) = b$, then the solution set to $L(v) = b$ is

$$\{v = v_p + v_H \text{ s.t. } L(v_H) = 0\}.$$
$$= \{v = v_p + v_H \text{ s.t. } v_H \in \ker(L)\}.$$

(page 4 Wed)

the "H" stands for "homogeneous sol'n", since $L(v) = 0$ is called the homogeneous equation

after page 5 Wed, proceed to page 2 today...

<u>Theorem</u> 4: Let $L(y)$ be the n^{th} order linear operator defined on page 1.
Then the dimension of $\ker(L) = n$.

(so if we can find n linearly ind. sol'ns to $L(y)=0$, they will be a basis for $\ker(L)$, i.e. every soln will be a linear combo of the basis sol'ns,

$$y_H = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \Big)$$

<u>proof</u>:

If $y(x) \in V$

write $\quad \vec{IV}(y) := \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}$

By the $\exists!$ theorem for IVP, $\exists$ sol'ns $y_1, y_2, \dots y_n$ to $L(y)=0$

such that

$$\vec{IV}(y_j) = \vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{entry}_j$$

Show $\{y_1, \dots y_n\}$ are a basis:

- These sol'ns are <u>linearly independent</u>, because if

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n \equiv 0 \qquad (\forall x \in I)$$

$$\frac{d}{dx} \Rightarrow \quad c_1 y_1' + c_2 y_2' + \dots + c_n y_n' \equiv 0$$

$$\frac{d^2}{dx^2} \Rightarrow \quad c_1 y_1'' + c_2 y_2'' + \dots + c_n y_n'' \equiv 0$$

$$\vdots$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} \equiv 0$$

called the Wronskian matrix of $y_1 \dots y_n$; its det is called Wronskian

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \qquad \forall x \in I$$

shows linear independence.

In particular, at $x = x_0$, the Wronskian matrix (above) is the identity, so $\vec{c} = \vec{0}$ ∎

- These solutions <u>span</u> $\ker(L)$ because if $z(x)$ solves $L(z)=0$, then

for $\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} := \vec{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$, the (other?) sol'n $y(x) = b_0 y_1(x) + b_1 y_2(x) + \dots + b_{n-1} y_n(x)$

has the same initial value vector!, i.e. $\vec{IV}(y) = \vec{b}$.

So, by uniqueness (in $\exists!$ theorem), $z \equiv y$! ∎

example: Let $L(y) = y'' + y' - 30y$

Find the $\ker(L)$, i.e. the general sol'n to $y'' + y' - 30y = 0$

hint: to find a potential basis try $y = e^{rx}$

Show linear ind. of you 2 sol'ns y_1, y_2 to show they are a basis.
(Hint: you could show the Wronskian (det) is non-zero)

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motivation to try $y = e^{rx}$ for any $\underbrace{\text{constant coefficient}}_{a_j(x) \equiv a_j \text{ const}}$ linear homogeneous DE:

In the example above,
if we write $D(y) := \frac{dy}{dx}$
$I(y) := y$ (identity)

then the L above can be written as

$$L = D^2 + D - 30I = (D + 6I)(D - 5I) = (D - 5I)(D + 6I)$$

["multiplication" of operators means composition here!]

and e^{-6x} solves $(D + 6I)\, y = 0$
e^{5x} solves $(D - 5I)\, y = 0$

this principle holds in general