Math 2280-1
Wed Sept 17

- A proof justifying our stability/instability claims from phase portrait analysis, is page 6 of today's notes.
- Then, do these notes, which are 3.1–3.2

Linear differential equations of n^{th} order

Def: $L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$

is called a linear differential operator of order n
We assume the functions $a_j(x)$, $0 \le j \le n$ are defined and continuous for $x \in I$, some interval in $\mathbb{R}$.

Theorem 1 L is a linear transformation
from the vector space of n-times continuously differentiable functions $y(x)$ defined for $x \in I$ ("V")
with codomain equal to the continuous functions on I ("W")

proof: This amounts to showing

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad \forall y_1, y_2 \in V$$
$$L(cy_1) = cL(y_1) \qquad c \in \mathbb{R}$$

or more compactly,

$$L(c_1y_1 + c_2y_2) = c_1 L(y_1) + c_2 L(y_2)$$

proof:

examples: $L(y) = y' + p(x)y$ $\quad 1^{st}$ order
$L(y) = y'' + p(x)y' + q(x)y$ $\quad 2^{nd}$ order

Def Let the $a_j(x)$, I be as on page 1.
Then an n^{th} order linear differential eqtn is an DE of the form

$$L(y) = f(x)$$

where f is continuous on I.

e.g.
1^{st} order: $y' + p(x)y = q(x)$
2^{nd} order: $y'' + p(x)y' + q(x)y = f(x)$

etc.

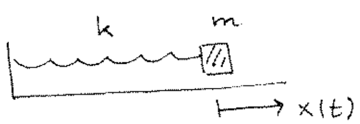
Theorem 2: Let $I \subset \mathbb{R}$ an interval
Let $x_0 \in I$
Let $a_0(x), a_1(x), \dots a_{n-1}(x), f(x)$ continuous on I
Then for any $b_0, b_1, \dots b_{n-1} \in \mathbb{R}$
$\exists!$ solution to the initial value problem

$$\begin{cases} L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x) & \forall x \in I \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

(We'll discuss proof later in course)

Examples: $n=1$: $\begin{cases} y'(x) + p(x)y = q(x) \\ y(x_0) = y_0 \end{cases}$ We did this §1.5 !!

$n=2$ example: oscillating spring IVP



$$m x'' = \text{net forces} = -\underset{\substack{\uparrow \\ \text{Hooke's const}}}{k}x - \underset{\substack{\uparrow \\ \text{coeff of fric}}}{c}x' + \underset{\substack{\uparrow \\ \text{external force}}}{f(t)}$$

$$\begin{cases} x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{1}{m}f \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

if we specify initial displacement and velocity of spring configuration we expect unique sol'n t for $t > 0$
(repeatable experiment!)

plausibility argument for Theorem 2:

Assume all functions are ∞'ly diff'ble and that $y(x)$ has a convergent

$\downarrow$

$a_j(x), f(x), y(x)$

Taylor series at x_0,

$$y(x) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{y^{(m)}(x_0)}{m!}(x-x_0)^m$$

From IVP we know coeffs at start of series

$y(x_0) = b_0$

$y'(x_0) = b_1$

$\vdots$

$y^{(n-1)}(x_0) = b_{n-1}$

from DE, $y^{(n)}(x_0) = f(x_0) - a_0(x_0)b_0 - a_1(x_0)b_1 - \cdots - a_{n-1}(x_0)b_n$

If we take $\frac{d}{dx}$ of DE, we can then solve for $y^{(n+1)}(x_0)$

then $\frac{d^2}{dx^2}$ of DE to get $y^{(n+2)}(x_0)$

recursively get all derivs of $y(x)$ at $x=x_0$, so the Taylor series for $y(x)$. !

Theorem 3 (Linear algebra)

Let $L: V \to W$ be a linear transformation.

Consider the problem of finding all solutions $\vec{v} \in V$ to the equation

$$L\vec{v} = \vec{b}$$

Let $\vec{v}_p$ be a (particular) solution to this equation.

Then every solution $\vec{v}$ can be written as

$$\vec{v} = \vec{v}_p + \vec{v}_H$$

where $\vec{v}_H$ is in the kernel of L, (i.e. solves the homogeneous equation $L\vec{v}_H = \vec{0}$)

(and each such $\vec{v} := \vec{v}_p + \vec{v}_H$ solves the original problem!)

proof: Let $\vec{v} = \vec{v}_p + \vec{v}_H$ where $L(\vec{v}_p) = \vec{b}$ and $L(\vec{v}_H) = \vec{0}$

Then $L(\vec{v}) = L(\vec{v}_p + \vec{v}_H)$

$= L(\vec{v}_p) + L(\vec{v}_H) = \vec{b} + \vec{0} = \vec{b}$, so $\vec{v}$ is a sol'n.

Now let $\vec{w}$ be any sol'n, i.e. $L(\vec{w}) = \vec{b}$

Then $\vec{w} = \vec{v}_p + \underbrace{(\vec{w} - \vec{v}_p)}_{= \vec{v}_H}$; note $L(\vec{w} - \vec{v}_p) = L(\vec{w}) - L(\vec{v}_p) = \vec{b} - \vec{b} = \vec{0}$

so $\vec{w}$ is the sum of $\vec{v}_p$ with a vector in the kernel of L ∎

example 1: $L: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Solve $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 1 & -2 & 3 & 1 \\ \hline 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = s$
$x_1 = 2s - 3t + 1$

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{"}\vec{x}_p\text{"}} + \underbrace{s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{"}\vec{x}_H\text{"}}$$

example 2: $L(y) := y' + 2y$

solve $y' + 2y = 3$ for $y(x)$, $x \in \mathbb{R}$

$$(e^{2x}y)' = e^{2x}3$$

$$e^{2x}y = \frac{3}{2}e^{2x} + C$$

$$y = \underbrace{\frac{3}{2}}_{\text{"}y_p\text{"}} + \underbrace{Ce^{-2x}}_{\text{"}y_H\text{"}}$$

For higher order DE's we will try to find the general solution by finding a y_P, and the general homogeneous sol'n

<u>example 3</u> Explain why <u>all</u> sol'ns to

$$y'' - 4y = 1 \qquad (\text{for } x \in \mathbb{R})$$

are of the form

$$y(x) = \frac{1}{4} + c_1 e^{2x} + c_2 e^{-2x}$$

<u>hint</u>: after finding y_P, and at least part of y_H, use $\exists!$ theorem to show every sol'n has claimed form.

Phase portrait justification theorem

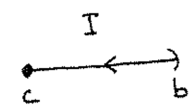
(because math majors deserve to understand!)

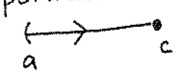
Consider the autonomous IVP

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(t_0) = x_0 \end{cases}$$

Assume f, f' are continuous $\forall x$, so that $\exists !$ theorem holds.

Let c be an equilibrium soln, i.e. $f(c) = 0$ (const.)

<u>Theorem 1</u> Let $f(x) < 0$ on the interval $\overset{I}{c < x < b}$, i.e. phase portrait 

or $f(x) > 0$ on the interval $\underset{J}{a < x < c}$, i.e. phase portrait 

Let the initial value $x_0 \in I$ or J.

Then $\lim_{t \to \infty} x(t) = c$

<u>proof</u>: (for the first case, i.e. the interval I.)

$x(t_0) = x_0 \in I$. $\not\exists\ t_1 {}^{\neq t_0}$ s.t. $x(t_1) = c$ since this would violate $\exists !$ theorem.

Thus (since $x'(t) = f(x) < 0 \Rightarrow x(t)$ dec.),

$$c < x(t) < x(t_0) \quad \forall\ t \geq t_0$$

Since $x(t)$ is decreasing and bounded below, (You did this for sequences in Calc.) (used bounded monotone)

$\lim_{t \to \infty} x(t) = L$ exists.

Let $n \in \mathbb{N}$, $n > t_0$. since $x(t) > c$, $L \geq c$

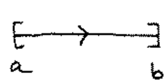
$$\frac{x(n+1) - x(n)}{1} = x'(t_n) \qquad n < t_n < n+1 \qquad \text{Mean Value Theorem!}$$

$$= f(x(t_n))$$

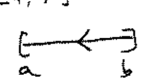
$$\downarrow n \to \infty \qquad \downarrow n \to \infty$$

$$0 = \frac{L - L}{1} = f(L) \qquad (\text{since } x(t_n) \to L \text{ and } f \text{ is continuous})$$

thus $f(L) = 0$. Thus L must equal c, since $c \leq L < x_0 < b$ ∎

<u>Theorem 2</u> 

$$\left.\begin{array}{l} f > 0\ \forall x \in [a,b] \\ x(t_0) \in [a,b] \end{array}\right\} \Longrightarrow \exists\ t_1 > t_0 \text{ s.t. } x(t_1) = b$$

(and analogous for ).

<u>proof</u>: $x(t)$ is increasing. If $x(t) < b$ $\forall\ t > t_0$ then $\lim_{t \to \infty} x(t) = L \leq b$ exists
(as long as $x(t) \in [a,b]$)

and $f(L) = 0$, as in theorem 1

this contradicts $f > 0$ $\forall x \in [a,b]$ ∎