Math 2280-1 and 2250-1
Friday September 12
Sinusoidal function essentials

These notes explain how to rewrite the linear combination
$$\mathrm{F}(t) = A\cos(\omega t) + B\sin(\omega t)$$
in amplitude-phase form
$$\mathrm{F}(t) = C\cos(\omega t - \alpha)\,.$$

The text has a version of this discussion in the section on Mechanical Vibrations (5.4 for 2250 text, 3.4 for 2280 text), and you will need to become proficient with these ideas. The current Maple project is a great time to begin! Here we assume C is a positive number, and use the following words for the various constants:

C is called the amplitude of $\mathrm{F}(t)$, since $\mathrm{F}(t)$ varies between C and $-C$.

ω is the angular frequency (units are radians per time, since the product ωt has radian units, so that cosine can eat it).

α is the phase angle (The graph of $\mathrm{F}(t)$ is obtained by translating of the graph of $C\cos(\omega t)$, by $\delta = \frac{\alpha}{\omega}$ units in the positive direction, since $\omega t - \alpha = \omega\left(t - \frac{\alpha}{\omega}\right)$. δ is called the time delay.

The frequency ν of $\mathrm{F}(t)$ is $\nu = \frac{\omega}{2\pi}$ (complete) cycles per unit time, since there are 2π radians per cycle.

The period $T = \frac{2\pi}{\omega}$ is the time it takes per unit cycle.

If you're given the amplitude-phase form of F it is straightforward to use the cosine angle addition (or subtraction) identities below to deduce the linear combination form. That the two identities below are equivalent follows easily from the facts that cosine is an even function and sine is an odd function.

$$\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$
$$\cos(z - w) = \cos(z)\cos(w) + \sin(z)\sin(w)\,.$$

Using the second version of the identity on the phase-amplitude form, and equating the result to the linear combination form yields
$$C\cos(\omega t - \alpha) = C\left(\cos(\omega t)\cos(\alpha) + \sin(\omega t)\sin(\alpha)\right)$$
$$= A\cos(\omega t) + B\sin(\omega t)\,.$$

We can make these two expression equal if (and only if) the constant coefficients of the functions $\cos(\omega t)$ and $\sin(\omega t)$ agree, i.e.
$$A = C\cos(\alpha)$$
$$B = C\sin(\alpha)$$
So, it's easy to get A and B if you know C and α. But the reverse direction is easy too, provided you're good at trigonometry: These two equations say that the point (A,B) in the Euclidean plane lies at a distance C from the origin, and at a polar-coordinate angle α from the positive x-axis. (Please see the triangle picture in section 5.4 or 3.4.) The fact that the distance from point (A,B) to the origin equals C

is just the identity

$$A^2+B^2=C^2\,(\sin(\alpha)^2+\cos(\alpha)^2)\ \ =C^2$$

so that

$$C=\sqrt{A^2+B^2}\ .$$

And then the equations above for A and B can be rewritten as

$$\cos(\alpha)=\frac{A}{C}$$

$$\sin(\alpha)=\frac{B}{C}$$

and these two equations determine the angle α (up to an integer multiple of $2\,\pi$). Depending on which quadrant the point (A/C,B/C) lies in, you may use different inverse-trig functions to find α. For example,

$$\alpha=\arccos\left(\frac{A}{C}\right),\ \ \text{in quadrants I,II (B>0)}$$

$$\alpha=\arcsin\left(\frac{B}{C}\right),\ \ \text{in quadrants I,IV (A>0).}$$

$$\alpha=\arctan\left(\frac{B}{A}\right),\ \ \text{in quadrant I (A,B>0).}$$

$$\alpha=\arctan\left(\frac{B}{A}\right)+\pi,\ \ \text{in quadrant III; (A,B<0).}$$

Example: Express $\mathrm{F}(t)=3\cos(2\,t)-4\sin(5\,t)$ in amplitude-phase form.

Answer: $A=3$ and $B=-4$, so the amplitude is $C=5$. Since the point (A,B) is in quadrant IV, we compute

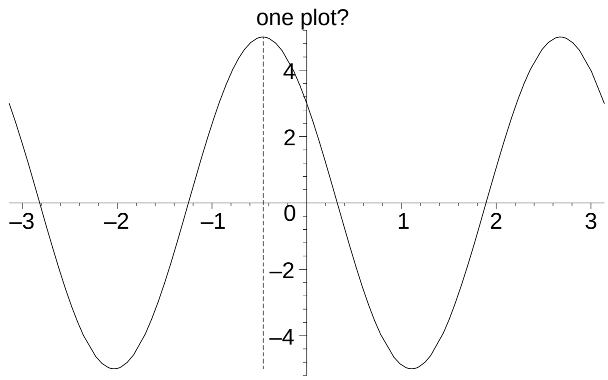
$$\alpha=\arcsin\left(\frac{B}{C}\right)\ =-0.927\ \ \text{radians.}$$

Thus, (allowing for decimal approximation),

$$3\cos(2\,t)-4\sin(2\,t)=5\cos(2\,t+0.927).$$

Below is graphical proof that we're right: notice that the horizontal translation (time delay δ) of the cosine curve is $\frac{\alpha}{2}=-0.46$ radians, i.e. the curve is translated to the left by .46 units. The display below includes two (completely overlapping plots of F(t), using the two different formulas, together with a dotted vertical line at t=-.46 radians.

```
> with(plots):
  plot1:=plot(3*cos(2*t)-4*sin(2*t),t=-Pi..Pi,color=black):
  plot2:=plot(5*cos(2*t+.927),t=-Pi..Pi,color=red):
  plot3:=plot([-.46,t,t=-5..5],linestyle=2,color=black):
  display({plot1,plot2,plot3},title='one plot?');
```




```
> ?plot  #help window to help me figure out to draw
         # the dotted line above, to show graph translation
```