

Solutions!

Name.....

I.D. number.....

Math 2280-1
Practice Second Midterm
(modified exam 2 spring 2006)

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin.
Good Luck!

1) Let A be the matrix

$$A := \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

1a) Find the eigenvalues and eigenvectors of A. (Hint: you should get 0 and -3 for the eigenvalues.)

(10 points)

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 - 2 = \lambda^2 + 3\lambda = \lambda(\lambda + 3) = 0 \text{ for } \boxed{\lambda = 0, -3}$$

$$\lambda = 0: \begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\boxed{\lambda = 0} \\ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -3: \begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \\ \hline 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\boxed{\lambda = -3} \\ \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

1b) Find the general solution to

$$e^{\lambda t} \vec{v} \text{ basis} \quad \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(5 points)

$$\boxed{\vec{x}_H(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

1c) Find all solutions to the nonhomogeneous system of differential equations

$$* \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

(10 points)

Use the method of undetermined coefficients. (Hint: for a particular solution, guess a constant vector).

Note, because $\lambda=0$ is an eigenvalue, our guessing algorithm would say to try $x_p = t\vec{k}_1 + \vec{k}_2$. But in this case $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is in the range of $A = \text{span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$, so the constant guess works:

into $* \vec{x}_p = \vec{k}$

$$0 = A\vec{k} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$A\vec{k} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\begin{array}{cc|c} -2 & 2 & -4 \\ 1 & -1 & 2 \\ \hline 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}$$

$k_1 = t$
 $k_2 = 2+t$

e.g. $\vec{k} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ works. (others differ by some x_h)

so $\vec{x}(t) = \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\vec{x}_p} + c_1 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}_h} + c_2 e^{-3t} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{x}_h}$

1d) Find the matrix exponential $e^{(A)t}$ for the matrix A in this problem, using your work from parts 1a, 1b.

$$e^{At} = \Phi(t) \Phi(0)^{-1} = \begin{bmatrix} 1 & -2e^{-3t} \\ 1 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2e^{-3t} \\ 1 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$\uparrow \quad \uparrow$
 $\vec{x}_1(t) \quad \vec{x}_2(t)$

$$= \frac{1}{3} \begin{bmatrix} 1+2e^{-3t} & 2-2e^{-3t} \\ 1-e^{-3t} & 2+e^{-3t} \end{bmatrix}$$

(10 points)

1e) Find all solutions to 1c) using variation of parameters - I'd recommend using the matrix exponential formula which we derived by mimicing the algorithm for scalar first order linear DEs.

(10 points)

$$x' - Ax = f$$

$$e^{-At} (x' - Ax) = e^{-At} f$$

$$(e^{-At} x)' = e^{-At} f$$

$$** e^{-At} x = \int e^{-At} f dt$$

in our case $\vec{f} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is a constant,
so $\int e^{-At} \vec{f} dt = (\int e^{-At} dt) \vec{f}$

mult ** on left by e^{At} :

$$x = e^{At} \left[\frac{1}{3} t \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \frac{1}{3} e^{3t} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \right] \begin{bmatrix} 4 \\ -2 \end{bmatrix} + e^{At} \vec{c}$$

would be a mess to multiply out!

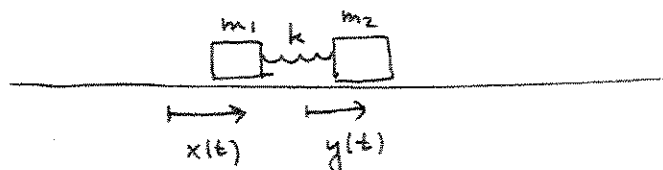
(But be prepared to derive the symbolic soln formula on an exam!)

alternate: definite integral

$$\int_0^t : e^{-At} x(t) - x_0 = \int_0^t e^{-As} f(s) ds$$

$$so x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-As} f(s) ds = \frac{1}{3} \begin{bmatrix} t + \frac{2}{3} e^{3t} & 2t - \frac{2}{3} e^{3t} \\ t - \frac{1}{3} e^{3t} & 2t + \frac{1}{3} e^{3t} \end{bmatrix} + C_{2 \times 2}$$

2) Consider a mass-spring system consisting of two masses coupled together with one spring and sliding along a frictionless plane, as indicated in the sketch below.



2a) Derive the second order system of differential equations which governs the motion of this system, for the two displacement functions $x(t)$, $y(t)$.

(4 points)

$$\begin{aligned} m_1 x'' &= k(y - x) \\ m_2 y'' &= -k(y - x) \end{aligned}$$

2b) What is the dimension of the solution space to this system of differential equations?

(2 points)

dim = 4 since this system is equivalent to a ~~4th order~~ ^{homog. linear} system of 4 1st order DE's,

$$\left(\begin{array}{l} \text{i.e.} \\ x_1 = x \\ x_2 = x' \\ y_1 = y \\ y_2 = y' \end{array} \quad \begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ +\frac{k}{m_1}(y_1 - x_1) \\ y_2 \\ -\frac{k}{m_2}(y_1 - x_1) \end{bmatrix} \right)$$

2c) Suppose the spring constant is 4 Newtons/meter, that $m_1 = 2$ kg and that $m_2 = 4$ kg. Show that in this case your general system from 2a) can be rewritten as

(4 points)

$$\begin{bmatrix} \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

↕

substitute into
2a):

$$\left. \begin{aligned} 2x'' &= 4(y-x) \\ 4y'' &= -4(y-x) \end{aligned} \right\} \Leftrightarrow \begin{aligned} x'' &= -2x + 2y \\ y'' &= x - y \end{aligned}$$

2d) Find the general solution to the system in (2c). Notice that the matrix is exactly the one you used in problem (1), so you already know its eigenvalues and eigenvectors from (1a). (Hint: For the case $\omega = 0$ you may need to think about the train motion in order to find a second linearly independent solution!)

(10 points)

$$\begin{array}{ll} \lambda = 0 & \lambda = -3 \\ \omega = \sqrt{-\lambda} = 0 & \omega = \sqrt{3} \\ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{array}$$

$$\vec{x}_H(t) = \underbrace{(c_1 + c_2 t)}_{\substack{\text{moving at} \\ \text{a constant} \\ \text{speed, in phase}}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{(c_3 \cos \sqrt{3} t + c_4 \sin \sqrt{3} t)}_{\text{oscillating out of phase}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

($\cos 0t = 1$)
note, if $A\vec{v} = \vec{0}$
and $\vec{x}(t) = (c_1 + c_2 t)\vec{v}$
then $\vec{x}''(t) = \vec{0} = A(c_1 + c_2 t)\vec{v}$
since $A\vec{v} = \vec{0}$

3) For the matrix

$$B := \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Maple says

`> eigenvectors(B);`

`[2 + I, 1, {[1, I]}], [2 - I, 1, {[1, -I]}]`

Use this information to find a basis of real vector-valued functions, for the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(10 points)

$$e^{(2+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{2t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$\operatorname{Re} e^{\lambda t} \vec{v}$, $\operatorname{Im} e^{\lambda t} \vec{v}$ are a basis, i.e.

$$\left\{ e^{2t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \right\}$$

4) Find the matrix exponential $e^{(C)t}$, for

$$C := \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

(Hint: you will either need to use chains or the fact that $C=2I+N$.)

(10 points)

$$C = 2I + N, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{(2I+N)t} = e^{2It} e^{Nt} \quad (e^{A+B} = e^A e^B \text{ when } A \& B \text{ commute})$$

$$= (e^{2t} I) \left(I + Nt + 0 \right)$$

↑
all higher order terms in power series are zero because $N^2=0$

$$= e^{2t} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$$e^{Ct} = \begin{bmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

alternak: $|C - \lambda I| = (A - 2)^2$.

$\lambda = 2$ evect is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$\text{FMS: } \begin{bmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

; since $\Phi(0) = I$,

$$e^{At} = \Phi(t) \Phi(0)^{-1} = \Phi(t)!$$

for other basis sol'n construct chain:

$$[C - 2I]^2 = N^2 = 0 \text{ so}$$

$$\text{let } \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{u} = [C - 2I] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{2t} (t \vec{u} + \vec{v}) = e^{2t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

5) Consider the forced damped oscillator

$$y'' + 2y' + 5y = \cos(t)$$

5a) Find the general solution to this differential equation

(5 points)

$$y_H: y = e^{rt}: \quad r^2 + 2r + 5 = 0$$

$$(r+1)^2 + 4 = 0$$

$$(r+1)^2 = -4$$

$$r+1 = \pm 2i$$

$$r = -1 \pm 2i$$

$$e^{(-1+2i)t}$$

$$= e^{-t}(\cos 2t + i \sin 2t)$$

$$y_H(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$$

thus $y = y_p + y_H$

$$= \frac{1}{5} \cos t + \frac{1}{10} \sin t + c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$$

y_p try

$$+ 5(y_p = A \cos t + B \sin t)$$

$$+ 2(y_p' = -A \sin t + B \cos t)$$

$$1(y_p'' = -A \cos t - B \sin t)$$

$$L y_p = \cos t [5A + 2B - A] + \sin t [5B - 2A - B] \stackrel{\text{want}}{=} \cos t [1]$$

$$\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 4A + 2B = 1 \\ -2A + 4B = 0 \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{10} \end{bmatrix}$$

5b) Exhibit an equivalent first order system of two differential equations, so that the first component function $x_1(t)$ of solutions to the system corresponds to solutions $y(t)$ to the differential equation in this problem.

(5 points)

$$(y =) \quad x_1' = x_2$$

$$(y' =) \quad x_2' = -2x_2 - 5x_1 + \cos t$$

, or in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

5c) Use your work in 5a) to exhibit the general solution to the system in 5b).

(5 points)

just take $\begin{bmatrix} y \\ y' \end{bmatrix}$ from the general soln to 5a)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} \cos 2t \\ -\cos 2t + 2 \sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin 2t \\ -\sin 2t + 2 \cos 2t \end{bmatrix}$$