Name...

I.D. number...

Math 2280-1
Practice Second Midterm
(modified exam 2 spring 2006)

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Let A be the matrix

$$A := \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$$

1a) Find the eigenvalues and eigenvectors of A. (Hint: you should get 0 and -3 for the eigenvalues.)

(10 points)

1b) Find the general solution to

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(5 points)

1c) Find all solutions to the nonhomogeneous system of differential equations

$$\begin{bmatrix} \dfrac{dx}{dt} \\[2mm] \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

(10 points)

Use the method of undetermined coefficients. (Hint: for a particular solution, guess a constant vector).

1d) Find the matrix exponential $e^{(At)}$ for the matrix A in this problem, using your work from parts 1a,1b.

(10 points)

1e) Find all solutions to 1c) using variation of parameters - I'd recommend using the matrix exponential formula which we derived by mimicking the algorithm for scalar first order linear DEs.

(10 points)

2) Consider a mass-spring system consisting of two masses m1 and m2, coupled together with one spring with spring constant k, and sliding along a frictionless plane. (think of two train coupled train cars.) Assume no external forces are being applied.

2a) Derive the second order system of differential equations which governs the motion of this system, for the two functions x(t), y(t) which measure displacement of the two cars from equilibrium.

(4 points)

2b) What is the dimension of the solution space to this sytem of differential equations?

(2 points)

2c) Suppose the spring constant is 4 Newtons/meter, that $m_1 = 2$ kg and that $m_2 = 4$ kg. Show that in this case your general system from 2a) can be rewritten as

(4 points)

$$\begin{bmatrix} \dfrac{d^2 x}{dt^2} \\[2ex] \dfrac{d^2 y}{dt^2} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

2d) Find the general solution to the system in (2c). Notice that the matrix is exactly the one you used in problem (1), so you already know its eigenvalues and eigenvectors from (1a). (Hint: For the case $\omega = 0$ you may need to think about the train motion in order to find a second linearly independent solution!)

(10 points)

3) For the matrix

$$B := \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Maple says

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> eigenvectors(B);
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$$[2 + I, 1, \{[1, I]\}], [2 - I, 1, \{[1, -I]\}]$$

Use this information to find a basis of real vector-valued functions, for the system

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(10 points)

4) Find the matrix exponential $\mathbf{e}^{(Ct)}$, for

$$C := \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

(Hint: you will either need to use chains or the fact that C=2I+N.)

(10 points)

5) Consider the forced damped oscillator
$$y'' + 2\,y' + 5\,y = \cos(t)$$
5a) Find the general solution to this differential equation

(5 points)

5b) Exhibit an equivalent first order system of two differential equations, so that the first component function $x_1(t)$ of solutions to the system corresponds to solutions $y(t)$ to the differential equation in this problem.

(5 points)

5c) Use your work in 5a) to exhibit the general solution to the system in 5b).

(5 points)